Platonic Solids and Graphs

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Abstract

This document explains what the platonic solids are and proves that there only exists only 5 platonic solids. Also shows duality of the five platonic solids and their dual pair of each platonic solid. The ways to prove the Euler characteristic on different solids. Form connected graphs on both plane and inner tubes and then find out the patterns to reflect the Euler characteristic of the surfaces.

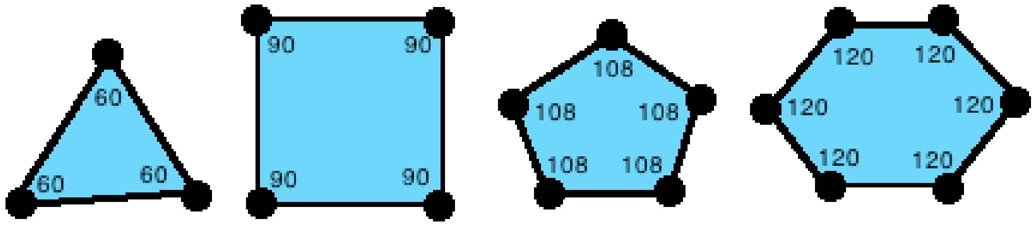
1 Definitions

A **Polygon** is a figure in the Euclidean plane consisting of a number of vertices and an equal number of line segments that connect vertices.
A **Regular Polygon** is a n-sided polygon where all the sides are the same length and each angle is equivalent.
A **Polyhedron** is a 3 dimensional solid consisting of a collection of polygons connected by their edges.
A **Simple Polyhedron** is a polyhedron that is topologically equivalent to a sphere (i.e., if it were inflated, it would produce a sphere) and whose faces are simple polygons.
A **Convex Polyhedron** is a polyhedron so that a line connecting noncoplanar points on the surface lies in the interior of the polyhedron.
A **Regular Polyhedron** is a convex polyhedron in which all faces are congruent and are regular polygons and have the same number of faces meeting at each vertex. This is also known as the **Platonic Solids**.

2 Platonic Solids

Theorem 2.1. *There only exists 5 Platonic Solids.*

Proof. We can construct platonic solids using regular polygons. The regular polygons that we will consider will be the equilateral triangle, square, regular pentagon, regular hexagon.



We need a minimum of at least three faces meeting at each vertex. Notice, the sum of the angles that form a vertex must be less than 360 degrees. If the sum equals exactly 360 degrees, then the resultant shape is a flat plane at the vertex, which means we will not be able to construct a platonic solid.

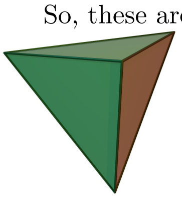
An equilateral triangle has internal angles of 60 degrees. Using the criterion above, we can have: 3 equilateral triangles meet at a vertex ($3*60 = 180$ degrees), 4 equilateral triangles meet at a vertex ($4*60 = 240$ degrees), 5 equilateral triangles ($5*60 = 300$ degrees).

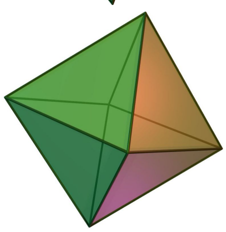
A square has internal angles of 90 degrees. We can have 3 squares meeting at a vertex ($3*90 = 270$ degrees).

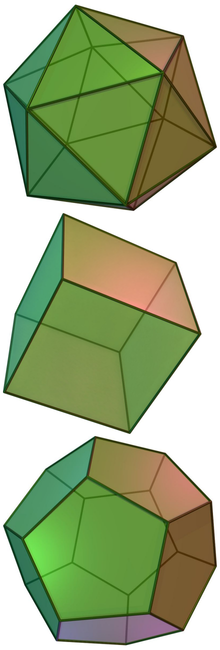
A regular pentagon has internal angles of 108 degrees. We can have 3 pentagons meet at a vertex ($3*108 = 324$ degrees)

A regular hexagon has internal angles of 120 degrees. If we have 3 regular hexagons meet at a vertex ($3*120 = 360$ degrees), it would not work since the sum of the angles at one vertex equals 360. So a regular hexagon and any other cannot be used to construct a platonic solid. $\square$

So, these are all the possible platonic solids. Here are the results:





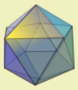


3 Duality of the Platonic Solids

Every polyhedron has a dual polyhedron, or in other words, the dual of every Platonic solid is another Platonic solid.

How to form a dual polyhedron from the original polyhedron? We could simply do that in two steps. First, place points on the center of every face. Second, connect the points in neighboring faces of the original polyhedron to obtain the dual. In such case, only interchange the number of faces and vertices while maintaining the number of edges.

Therefore, it is possible to arrange the five solids into dual pairs. The table below shows several pairs of dual platonic solids with vertices and faces interchanged but with the same edges.

Name		Faces	Vertices	Edges	F+V-E
Tetrahedron		4	4	6	**2**
Cube		6	8	12	**2**
Octahedron		8	6	12	**2**
Dodecahedron		12	20	30	**2**
Icosahedron		20	12	30	**2**

From the table, we can conclude that:

- The tetrahedron is self-dual (i.e. its dual is another tetrahedron).
- The cube and the octahedron form a dual pair.
- The dodecahedron and the icosahedron form a dual pair.

4 Euler characteristic

In this section we are going to prove the famous Euler characteristic. Each subsection we would focus on Euler characteristic on a different solid.

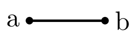
For a given polyhedron, let V be vertices, E be edges, and F be faces, Euler characteristic shows that

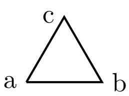
$$\chi = \#V - \#E + \#F = 2,$$

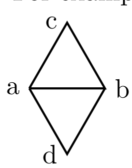
that is, the number of vertices minus the number of edges plus the number of faces is 2.

4.1 Cauchy

This method is devised by Cauchy. We would first establish two algorithms and then prove the Euler characteristic based on these two algorithms.

For a simple polyhedron with vertices V, edges E, and faces F, we define **degree** of an edge to be the number of faces nearby. For example, 

, degree(ab) = 0; 

, degree(ab) = 1; and 

has degree(ab) = 2;

From the previous examples, we can view degree as function:

$$\text{degree} : E \to \{0, 1, 2\}.$$

We call one edge e **free** when $degree(e) \neq 2$, intuitively, it means that e is exposed to outside.

And since the degree is either 0 or 1 or 2, $E = \{\text{free edges}\} \bigoplus \{\text{non-free edges}\}$.

Algorithm 1 Triangularization

Input: face $f \in F$
1: Pick any point p inside f
2: Let V be f's vertices
3: **for** $v \in V$ **do**
4:     connect p and v

Algorithm 2 Triangular Reduction

Take one face away
while $\#F > 1$ **do**
    for $f \in F$ **do**
        if f not a triangle **then**
            Triangularization(f)
        Take all free edges from f

If you have a polygon, and you randomly pick one point and connect the point with each vertices, you would end up with a polygon with many sub-triangle. Based on this intuition, we present the following lemma.

Lemma 4.1. *Algorithm 1 would not change χ.*

Proof. The input is one face of this simple polyhedron, note that it is not necessarily to be a polygon. Let assume it has vertices n, that is, $\#V = n$.

There are two cases:

1. p is not one of the vertices.

   Then line 1 would introduce one more vertex, thus χ increases by 1.

   For line 4, when p lies in one of the edges, it would split this edge into two, creating $n-1$ vertices and change one face to $n-1$ faces. Therefore $\Delta\chi = \Delta\#V - \Delta\#E + \Delta\#F = 1 - (n-1) + (n-2) = 0$.

   Otherwise, we are introducing n edges, and these n new edges break one face to n faces, thus $\#F$ increases by $n-1$. Therefore $\Delta\chi = \Delta\#V - \Delta\#E + \Delta\#F = 1 - n + (n-1) = 0$.

2. $p \in V$.

   Then line 1 is not adding more vertex and line 4 introduces $n-3$ edges, thus breaking one face to $n-2$ faces, thus $\Delta\chi = \Delta\#V - \Delta\#E + \Delta\#F = 0 - (n-3) + (n-3) = 0$.

In both cases, we show that χ doesn't change. ∎

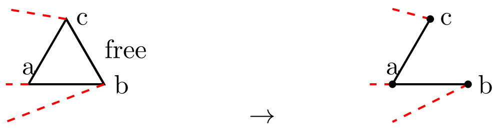
What if you keep doing the triangularization on each surface and remove the out-most edges? Then you would end up with one triangle, then calculate Euler characteristic is easy. The algorithm 2 is a way to reduce the complicated polyhedron to one much simpler polygon.

Lemma 4.2. *Besides line 1, Algorithm 2 does not alter χ.*

Proof. For line 5, by lemma 4.1, it would not change χ.

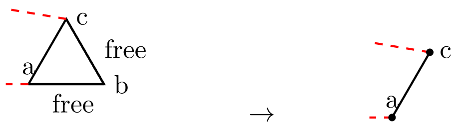
For line 6, there are three cases.

i one free edges.



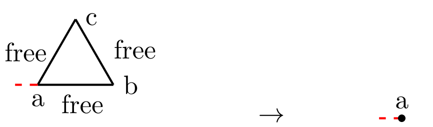
In this case, we lose one edge and one face but retain all vertices. Thus $\Delta\chi = \Delta\#V - \Delta\#E + \Delta\#F = 0 - 1 + 1 = 0$.

ii two free edges.



In this case, we lose two edges and the intersected vertices, and one face. $\Delta\chi = \Delta\#V - \Delta\#E + \Delta\#F = 1 - 2 + 1 = 0$.

iii no free edge



In this case, we lose all three edges and retain only one vertex. $\Delta\chi = \Delta\#V - \Delta\#E + \Delta\#F = 2 - 3 + 1 = 0$.

In all three cases the Euler characteristic remains the same. Therefore the while loop does not alter χ. □

Now we are ready to prove Euler characteristic!

Theorem 4.3. *Euler Characteristic holds on simple polyhedron.*

Proof. We can always do Algorithm 2 to the simple polyhedron since the polyhedron is simple, therefore there is no hole in its faces, thus triangularization is possible. In the line 1, we remove one face but keep others intact. Thus χ decrease by 1. But by lemma 4.2, besides line 1, it does not alter χ. Therefore until the Algorithm 2 stops, χ does not change, and the algorithm won't stop until the number of face becomes 1, which would be a triangle. Calculating the Euler characteristic of triangle is easy: $3-3+1 = 1$, thus χ for the whole simple polyhedron is the χ for a triangle, which is 1. Therefore, adding the face we remove in line 1, we have $\chi = 2$. □

4.2 Reverse thinking

We now present another method to prove Euler characteristic on a simple regular polygon. Opposite from Cauchy's, we are now *augment* surfaces instead of reducing them.

For a n-simple regular polygon, each face is a n-polygon. The intuition is that, when you are making a net of a simple regular polyhedron, you are build it from one single face and construct each surface surrounding this face.

Theorem 4.4. *Euler Characteristic holds on simple regular polyhedron.*

Proof. First we start with one arbitrary face, $\chi = \#V - \#E + \#F = n - n + 1 = 1$, and we then consecutively add one face at a time. Let m be the number of shared vertices when adding the new face.

When $1 \leq m < n$, after adding we are introducing $n - m$ new vertices, and $n - m + 1$ new edges since the new edges fully connect the $n - m$ new vertices. Therefore $\Delta\chi = \Delta\#V - \Delta\#E + \Delta\#F = (n - m) - (n - m + 1) + 1 = 0$.

When $m = n$, since all new edges are same as the polyhedron we are adding on, we are adding the last surface. In this case we are not introducing any new edges or vertices but merely providing one face. Therefore $\Delta\chi = \Delta\#V - \Delta\#E + \Delta\#F = 0 - 0 + 1 = 0$.

Overall we starts with $\chi = 1$, and after successfully adding all surfaces, χ increases by 1, therefore for the final simple regular polyhedron, $\chi = 2$. $\square$

4.3 Connected Planar Graph

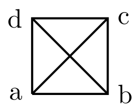
The Euler characteristic appears in the study of "connected planar graphs". What are those? And how to form connected planar graphs? Could we draw a connected graph on different surfaces? Do these new graphs still satisfy the Euler characteristic?

- ***Definition***

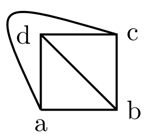
A **graph** means a finite collection of vertices or dots together with a finite collection of edges. In addition, the edges could be either straight or curve paths, as long as they are beginning and ending at the vertices.

A graph is called **connected** if each vertex is connected with another one, which means there is no isolated vertex on a graph.

A graph is called **planar** if the edges only meet each other at vertices. So there is no intersection on a graph.

For example, 

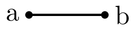
is a planar graph since it can be drawn in the following way.



Let $G =< V, E >$ be a connected planar simple graph with vertices V and edges E. To be more precise, $E \subseteq \{(x, y) \mid (x, y) \in V \times V \wedge x \neq y\}$.

A subgraph H of graph G is $H =< W, T >$ where $W \subseteq V$ and $E \subseteq T$.

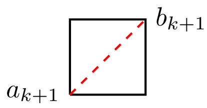
Theorem 4.5. *Euler Characteristic holds on a connected planar graph. Note in terms of planar graph, $\#F$ is the number of region.*

Proof. We construct a sequence of subgraphs G_1, G_2, ..., G_E by consecutively add one edge. First we arbitary pick one edge to obtain G_1, which looks like 

, and for $i = 1, 2, \ldots, E-1$, we obtain G_{i+1} by adding one edge that is incident with a vertex in G_i as well as other vertex incident with edge not in G_i. This construction is always possible since G itself is connected. And G is obtained after E edges are added, that is, $G_E = G$. For $n = 1, 2, \ldots, E$, let v_n, e_n, f_n be the number of vertices, the number of edges, the number of faces of G_n respectively.

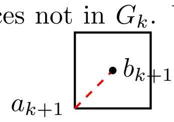
We want to show that for each n, that is, $v_n - e_n + f_n = 2$, we prove by induction on n. When $n = 1$, the previous graph shows that it is true since $v_1 - e_1 + f_1 = 2 - 1 + 1 = 2$. (recall that in terms of planar graph, $\#F$ is the number of region.)

Now assume for $1 < k < n$ it is true, $v_k - e_k + f_k = 2$, let (a_{k+1}, b_{k+1}) be the edge added into G_k. There are two cases,

i two vertices a_{k+1} and b_{k+1} are already in G_k. 

In this case the common region shared by these two vertices are split into half, and $v_{k+1} = v_k$, $e_{k+1} = e_k + 1$, and $f_{K+1} = f_k + 1$, therefore

$$\begin{aligned} v_{k+1} - e_{k+1} + f_{k+1} &= v_k - (e_k + 1) + f_k + 1 \\ &= v_k - e_k + f_k \\ &= 2. \qquad \text{(inductive hypothesis)} \end{aligned}$$

ii one of the vertices not in G_k. Without the loss of generality, we assume b_{k+1} not in G_k. 

In this case b_{k+1} must be in a region that has a_{k+1} as a boundary, thus not producing new region. $v_{k+1} = v_k + 1$, $e_{k+1} = e_k + 1$, and $f_{K+1} = f_k$, therefore

$$\begin{aligned} v_{k+1} - e_{k+1} + f_{k+1} &= (v_k + 1) - (e_k + 1) + f_k \\ &= v_k - e_k + f_k \\ &= 2. \qquad \text{(inductive hypothesis)} \end{aligned}$$

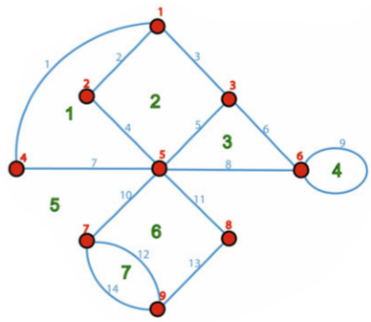
Therefore by mathematical induction for all n, $v_n - e_n + f_n = 2$. And G is graph G_E, after adding E edges we obtain G with the same characteristic. □

4.4 Form a connected graph on different surfaces

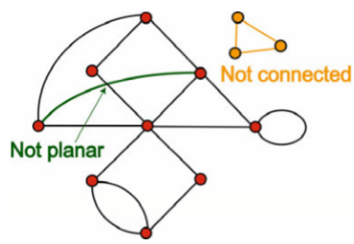
1. ***Form a connected planar graph***

For instance, Let's think of the vertices as the towns and the edges as roads between the towns. We have to ensure that it is possible to travel between any pairs of towns along the roads, and also that the roads might only meet each other at towns but do not otherwise cross each other.

Now, let's draw a connected planar graph and count the numbers of Vertices(V), Edges(E), and Faces(F) that the graph has. Notice that one of the faces is always unbounded. It shows that V-E+F=9-14+7=2



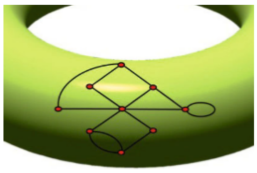
Now we have got a perfect connected planar graph. How to change it into a non-planar graph?
Adding a new edge that crosses another edge at a non-vertex location.
How to change it into a non-connected graph?
Adding a new cluster of edges and vertices with no bridge to the original cluster.



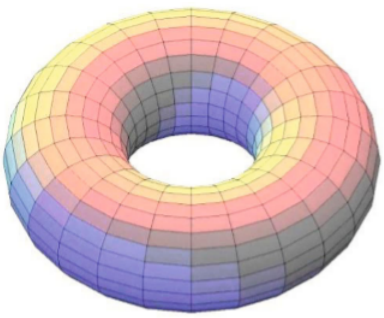
2. ***Graph on different surfaces***

Are there any settings in which Euler's formula does NOT work? Let's try to draw a connected graph on an inner tube or a "double inner tube" or a "triple inner tube" .

Should we expect the expression V + F − E= 2 as before? Actually if we use only a small part of the rubber surface, then we get the same V, E, and F counts when we drew the same graph on a piece of paper.



But the largest face(unbounded face or Face 5 in the previous connected planar graph) of this graph is very peculiar. Unlike the other faces, its shape could not be formed by deforming (bending or stretching) a rubber polygon. We have to start over and draw a new graph whose faces are all shaped like deformed polygons.



We carefully count V = 576 , E = 1,152 and F = 576 (faces are all deformed squares). Therefore, the expression V + F − E equals zero (not 2). In fact, the expression V + F − E equals zero for ANY connected graph embedded on an inner tube. Any such graph on the “double inner tube” will satisfy V + F − E = −2. Any such graph on the “triple inner tube” will satisfy V + F − E = −4. Different surfaces can be distinguished by their value of V + F − E. This value is called the Euler characteristic of the surface.