One probability puzzle of pointer

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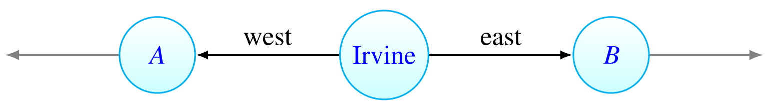
1 Predicate the coin flipped

1.1 Where am I?

You come back from a long trip. You are in LAX now, and you are going back to Irvine. Of course you take Flixbus, because it is cheap!

Now you are abroad, but it turns out that the bus driver does not even know whether he should drive west or east. Therefore, he can only do nothing but flip a coin, Head is West, Tail is East. Now to simplify the situation, we let each station only has two adjacent stations, one in the west while another in east.

What's worse, you fall asleep during the drive. After some time you wake up, you are totally lost. You only know you are heading to Irvine, but in which direction? Where am I? Am I in station A, or am I in station B?



1.2 Pointer

You realize that you have a pointer APP on your phone. The direction it points to should be where this bus is heading to. But you realize that due to the terrible GPS signal, the pointer is not 100% accurate. So you think of your pointer as a random continuous variable that spreads through all the stations. However, you decide to trust the pointer nevertheless.

Let

$$p \coloneqq \textbf{probability of pointer points to west of A},$$
$$q \coloneqq \textbf{probability of pointer points to east of B}.$$

Note $p + q < 1$ since there is also a gap in station A and B.

Since you are equal-likely to be in A and B, the probability of the pointer correctly pointing to the direction the train is heading to, that is, successfully guessing[1] the result of coin flipped, is

$$\frac{1}{2} \cdot (1 - p) + \frac{1}{2} \cdot (1 - q) > \frac{1}{2}.$$

[1]Note it is not the same as predicting the coin. The coin is already landed.

Once you guess where you are heading to, and know the next stop is Irvine, you can guess where you are right now! Therefore, it seems that although the exact values of q and p are arbitrary (Remember the GPS signal is terrible and the pointer is unreliable), if you put some faith on your broken GPS, you are still more likely to guess correct!

1.3 Why not make a friend in the bus?

Whoa, it turns out that there is another one in the bus who also has no clue what is going on. He says that he also made a guess. But he made the guess before the bus driver tossed the coin. So he is predicting[2] the result of the coin flipped.

What a coincide. He says he also has a pointer APP, and his prediction is based on the pointer as well. The GPS signal is terrible, his pointer is not accurate as well. But since he has the same information as you are, he says proudly, ”My prediction is more than 50% correct.”

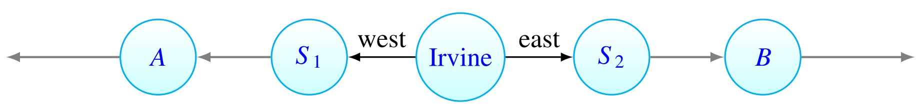
But not so fast! You contemplate, isn't the coin flipped always 50-50 chance Head or Tail? How come he can predict the outcome of the coin with more than 50% chance? Is it because we are not equal-likely to be in station A and B? No, says your new friend, it is impossible. [3] Then it is certain that there is ***something*** going on in this paradox.

2 Demystify

To illustrate this ***something***, or, a hidden variable, we present two experiments.

2.1 Experiment 1, with hidden variable

Now if the stations are located as



And, this time the bus driver would use a spinner with Red and Blue color. The probability that the spinner lands in the red region is r, which is greater than $\frac{1}{2}$. Red region means towards Irvine, while blue region means away from Irvine.

To see that the bus is still 50% chance heading to east, note the bus is equal-likely in station S_1 and S_2,

$$\frac{1}{2}r + \frac{1}{2}(1 - r) = \frac{1}{2}.$$

[2]Compared to guessing, the coin has not flipped yet.

[3]He says, ”Flipping a coin to decide next step is a random walk. A random walk on a linear track with two end stations is a Markov chain with reflecting barriers, and the steady state probabilities for all the stations in the middle are the same. Similarly, a random walk on a circular track with a finite number of stations has the same steady state probability for all stations. Also, one could simply mandate the assumption. So there is apparently justification for assuming that a destination can have two equally probable station of origin.”

Note that our unawareness of where we are makes the probability of bus heading to each direction is still 50%. So for the passanger there is no difference from the previous one.

However, it is obvious that there is something hidden: at station S_1 the bus is more likely go east while in station S_2 the bus is more likely go west.

Now let

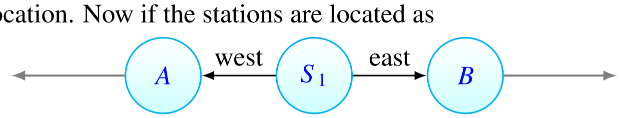
$$
\begin{aligned}
p_1 &:= \textbf{probability of pointer points to west of S1},\\
p_2 &:= \textbf{probability of pointer points in between S1 and S2},\\
p_3 &:= \textbf{probability of pointer points to east of S2}.
\end{aligned}
$$

Now the probability that the pointer direct the right direction is

$$
\begin{aligned}
&\overbrace{\frac{1}{2}r(p_2+p_3)}^{\text{spinner Red and Station S1}} + \overbrace{\frac{1}{2}(1-r)p_1}^{\text{spinner Blue and Station S1}} + \\
&\underbrace{\frac{1}{2}r(p_1+p_2)}_{\text{Spinner Red and Station S2}} + \underbrace{\frac{1}{2}(1-r)p_3}_{\text{Spinner Blue and Station S2}} = \frac{1}{2}(p_1+p_3) + rp_2 \\
&= \frac{1}{2} + (r-\frac{1}{2})p_2 \\
&> \frac{1}{2}.
\end{aligned}
$$

2.2 Experiment 2, without hidden variable

Note there are two way to interpret the information ”Next stop is Irvine”, one is as what we do so far, that we are in east of Irvine or we are in west of Irvine; another is what we will use in this experiment, that Irvine is either in east of our location or in west of our location. Now if the stations are located as



Let’s say the passenger is equal likely to be at station S_1 and S_2. Let passenger at station S_1. Irvine can be either in station A or B. And the bus driver flips a coin to decide the direction. Head is west and Tail is east.

Now let

$$
\begin{aligned}
q_1 &:= \textbf{probability of pointer points to west of S1},\\
q_2 &:= \textbf{probability of pointer points to east of S1},
\end{aligned}
$$

It is clear that the probability that $q_1 + q_2 = 1$, and the pointer points to the right direction is

$$
\overbrace{\frac{1}{2}q_1}^{\text{coin Head and pointer west}} + \overbrace{\frac{1}{2}q_2}^{\text{coin Tail and pointer east}} = \frac{1}{2}(q_1+q_2) = \frac{1}{2}.
$$

Only half the chance the pointer gets it right.

2.3 A Comparison of Experiments 1 and 2

From the standpoint of the passengers, these two experiments are the same. Even though in Experiment 1, the bus goes east from S_1 more often than it goes east from S_2, but there is no way for passengers to know. After all the passengers are just in a unknown location guessing the direction they are heading to. And in both experiment he only knows that the two outcomes, heading east & west, occur independently with probability $\frac{1}{2}$.

However, our broken pointer can ”detect” that the bus go somewhere more often, and thus by using the pointer, the passengers can ”know” whether there is a hidden variable involved. If he can predict with more than 50% accuracy, then his situation involves hidden variable, while if his prediction is only 50%, then his situation has no hidden variabel.

It is a paradox that depends on how you interpret the scenario you are at right now, the two scenarios you come up with can yield different results.

> Note, that empirical measurement contradicts the Principle of Indifference, which assumes that in absence of additional information, all outcomes are equally probable.