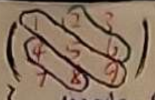
trace / Span

1. let $A \in M_n(F)$ 

- main diagonal $\{a_{ii}\}$, main subdiagonal $\{a_{i,i-1} \mid i=2,..,n\}$, main superdiagonal $\{a_{i-1,i} \mid i=2..,n\}$

Date ▪ trace $:= \sum a_{ii}$

$\sum(rA_{ii}+sB_{ii}) = r\sum A_{ii} + s\sum B_{ii}$

let $ABC \in M_n(F)$, $r, s \in F$, $tr(rA+sB) = r\,tr(A)+s\,tr(B)$ (linearity)

(commutative) $tr(AB)=tr(BA)$ $\sum_{i=1}^{n}(AB)_{ii} = \sum_i \sum_k A_{ik}B_{ki} = \sum_k \sum_i B_{ki}A_{ik} = \sum_k (BA)_{kk}$ OR by $\boxed{15}$. $P_{⑥}$ 和 $\boxed{24}$ 注 1

$tr(ABC)=tr(BCA)$ (Cyclic) $\sum_t (ABC)_{tt} = \sum_t \sum_k A_{tk}(BC)_{kt} = \sum_t \sum_k A_{tk} \sum_j B_{kj} C_{jt}$

$= \sum_k \sum_t \sum_j B_{kj} C_{jt} A_{tk} = \sum_k \sum_t BC_{kt} A_{tk} = \sum_k (BCA)_{kk}$

注意 一般 $tr(AB) \neq tr(A) + tr(B)$

$tr(A^T) = tr(A)$

$tr(A^TB) = \sum_i \sum_j A_{ij} B_{ij}$, $tr(u \otimes v) = u^Tv$ $tr\begin{pmatrix} u_1v_1 & & \\ \vdots & \ddots & \\ & & u_nv_n \end{pmatrix}$

2. 对 Vec. Space V 的 nonempty subset S. span(S) 为 L. comb. of vectors in S, 令 $span(\emptyset) = \{\overset{\in V}{0}\}$

Algebra: 这也是为何 span(S) 可记为 $\langle S \rangle_p$, 为 subspace, P11. 4 eg1 的[illegible], [illegible]而 set, 代数无 L.comb, 用 generator 表示

$\mathcal{T}_1$ 若 $S \subseteq V$, span(S) 为 V 的 subspace; 事实上, 若 V 的 subspace W contains S, 也 contains span(S)

☐ i) 若 $S=\emptyset$, $\{0\}$ 为任何 vector space 的 trivial subspace; 若 $S \neq \emptyset$, $\underset{\in F}{0}\underset{\in span(S)}{x} = 0 \in V$, 而 $x, y \in span(S)$ $x = a_1v_1 + .. + a_nv_n$, $y = b_1u_1 + .. + b_mu_m$ $a_i, b_i \in F$

易证 $x+y$, cx 皆为 span(S) 中

ii) 即: $s_1 \dots s_n \in W$, 对 $a_i \in F$, $\underline{a_1s_1 + .. + a_ns_n \in W}$ induction on n. 用 V.S. 的两 closure

3. DEF $S \subseteq$ vector space V. S generates/spans V iff span(S) = V

eg. 1. $\{(110)(101)(011)\}$ spans $\mathbb{R}^3$ ∵ 对 $\forall x = (abc) \in \mathbb{R}^3$, $(abc) = x(110)+y(101)+z(011)$ $x = \frac{a+b-c}{2}$ $y = \frac{a-b+c}{2}$ $z = \frac{-a+b+c}{2}$

$\mathcal{T}_2$: $W \leq V \Leftrightarrow span(W) = W$

☐ ⇒, $span(W) \subseteq W$: 因 $W \subseteq W$, $\underline{\overset{据 \mathcal{T}_1 ii)}{span(W) \subseteq W}}$; $span(W) \supseteq W$: let $w_i \in W$, $w_i = 0w_1 + .. + 1w_i + .. \in span(W)$

⇐ $span(\emptyset) = \{0\}$, ∴ W nonempty; $\forall w_1, w_2 \in W$, $w_1 + w_2 \in W$ (L.comb 相加 仍 L.comb); scalar multiplication trival

4. $\mathcal{P}$ i) (span 的传递) S_1, S_2 为 V 的 subset, $S_1 \subseteq S_2 \Rightarrow span(S_1) \subseteq span(S_2)$; 若 $span(S_1) = V$, $span(S_2) = V$

ii) $S_1, S_2 \subseteq V$, $span(S_1 \cup S_2) = span(S_1) + span(S_2)$

☐ ⊆: $e \in LHS$ 可写为 $\underbrace{\sum^n a_i s_{1i}}_{\in span(S_1)} + \underbrace{\sum^m b_i s_{2i}}_{\in span(S_2)}$. ⊇: $e \in RHS$ 可写为 $\underbrace{u}_{span(S_1)} + \underbrace{v}_{span(S_2)}$

iii) $S_1, S_2 \subseteq V$, $span(S_1 \cap S_2) \subseteq span(S_1) \cap span(S_2)$ 注意 非等号

☐ $e \in LHS$ 为 $\sum a_i s_i$ ($s_i \in S_1 \cap S_2$), ∴ $e \in span(S_1)$ $e \in span(S_2)$ 如 $V = \mathbb{R}^2$, $S_1 = \{(1,0)\}$, $S_2 = \{(0,1)\}$ 如P11.4.eg.1 $LHS = span(\emptyset) = \{(0,0)\}$ $RHS = \{(0,0)\}$, LHS=RHS

又: $S_1 = \{(1,0)(1\,1)\}$, $S_2 = \{(10)(01)\}$ $LHS = \langle(1,0)\rangle$, $RHS = \mathbb{R}^2$

iv) 对 $S \subseteq V$. 若 S L.Ind. 则 $\forall s \in \overset{即一L.Ind 的 span}{span(S)}$, s 可由 S unique 表示

☐ let $s = \sum a_i s_i = \sum b_i s_i$ $a_i b_i$ 不全相同, 则 $s - s = 0 = \sum (a_i - b_i) s_i$, 由 L.Ind, $a_i - b_i = 0, \forall i$ ∴ $a_i = b_i$

v) (消去) 对 $span(v_1 .. v_n)$ 若 v_i 可被其余 L.comb, 则 $span(v_1 .. v_n) = span(v_1 .. v_{i-1}, v_{i+1} .. v_n)$

注 1. $x^Ty \cdot y^Tx = tr(xx^T yy^T) = \sum_{i,j} x_i x_j \cdot y_i y_j$

☐ 令 $u = x$, $v = yy^Tx$ 由 1. $u^Tv = tr(\overset{A^T}{x} x^T y \overset{B}{y^T}) = \sum_{ij} (xx^T)_{ij} (yy^T)_{ij}$

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例1 求 $\det\left(\begin{bmatrix}1&2&3&\cdots&n\\2&3&4&\cdots&n&1\\3&4&5&\cdots&1&2\\\vdots\\n&1&2&\cdots&n-2&n-1\end{bmatrix}\right)$,

□ note 各行之和相同, ∴ 各列全加于 col 1, 提出 col 1: $=\sum_{i=1}^{n} i\begin{vmatrix}1&\cdots&n\\1&3\cdots&n&1\\\vdots&&&\vdots\\1&\cdots&&n-1\end{vmatrix}$ note 相邻行相差1, 依次上一行 ×(-1) 加于下一行

$=\sum i\begin{vmatrix}1&\cdots&&n\\0&1\cdots&1&1-n\\0&1\cdots&1-n&1\\\vdots\\0&1-n&\cdots&1\end{vmatrix}$ col 1 展开, $=\sum i\underbrace{\begin{vmatrix}1&\cdots&1-n\\\vdots&1-n&1\\1-n&\cdots&1\end{vmatrix}}\Big\}n-1$ note 各行之和 $=-1$ 依进一步, 各列全加于 col 1 $=\sum i\begin{vmatrix}-1&\cdots&\cdots&1-n\\-1&\cdots&1-n&1\\-1&1-n&&1\\-1&1&\cdots&1\end{vmatrix}$

用第一列的 -1 cancel 其余的 1 列 $\sum i\begin{vmatrix}-1&0&\cdots&-n\\\vdots&&\cdot^{\cdot^{\cdot}}\\-1&-n&\cdots&0\\-1&0&\cdots&0\end{vmatrix}$ last row 展开 $=\underline{\sum i(-1)^{n-1+1}}\cdot-1\cdot\overbrace{\begin{vmatrix}0&&-n\\&\cdot^{\cdot^{\cdot}}\\-n&&0\end{vmatrix}}^{n-2}=\frac{n(n+1)}{2}\cdot(-1)^{n+1}\cdot\left[(-1)^{n-2+\frac{(1+n-2)(n-2)}{2}}(-n)^{n-2}\right]$

例2 $\begin{vmatrix}a_1&a_2&\cdots&a_n\\b_2&1&&0\\\vdots&&\ddots\\b_n&0&&1\end{vmatrix}$ 各列运算后 加入 col 1 $\begin{vmatrix}*&a_2&\cdots&a_n\\0&1\\\vdots&&\ddots\\0&&&1\end{vmatrix}$ $*=a_1-a_2\cdot b_2-\cdots-b_na_n$ $=a_1-\sum_{i=2}^{n}a_ib_i$

例3 $\begin{vmatrix}a_1+b_1&a_1+b_2&\cdots&a_1+b_n\\a_2+b_1&a_2+b_2&\cdots&a_2+b_n\\&\vdots\\a_n+b_1&a_n+b_2&\cdots&a_n+b_n\end{vmatrix}=\begin{vmatrix}a_1&a_1+b_2&\cdots&a_1+b_n\\a_2\\\vdots\\a_n&a_n+b_2&\cdots&a_n+b_n\end{vmatrix}+\begin{vmatrix}b_1&a_1+b_2&\cdots&a_1+b_n\\b_1\\\vdots\\b_1&a_n+b_2&\cdots&a_n+b_n\end{vmatrix}$ 用各列去掉 col 1 $=\begin{vmatrix}a_1&b_2&\cdots&b_n\\a_2&b_2&\cdots&b_n\\\vdots&\vdots\\a_n&b_2&\cdots&b_n\end{vmatrix}$ × $\frac{b_2}{b_n}$ 后与 col 2, col n 相同, det = 0

第i列 $-\frac{b_i}{b_1}$ 第1列 i col $+\begin{vmatrix}b_1&a_1&\cdots&a_1\\b_1&a_2&\cdots&a_2\\\vdots\\b_1&\underline{a_n}&\cdots&\underline{a_n}\end{vmatrix}=0+0=0$ 两 col 相同

例4 $\begin{vmatrix}a_1&b_1&0&\cdots&0\\c_1&a_2&b_2&\ddots&\vdots\\0&c_2&\ddots&\ddots\\\vdots&&\ddots&&b_{n-1}\\0&\cdots&\cdots&c_{n-1}&a_n\end{vmatrix}=$ □ 令其为 D_n, 对 last row expand, $=a_nD_{n-1}-c_{n-1}\begin{vmatrix}a_1&&0\\&\ddots&\vdots\\0&\cdots c_{n-2}&b_{n-1}\end{vmatrix}$

expand last col

$=a_nD_{n-1}-c_{n-1}\cdot\underline{b_{n-1}}\cdot D_{n-2}$

例5 $\begin{vmatrix}x&y&\cdots&y\\z&\ddots&&\vdots\\\vdots&&\ddots&y\\z&\cdots&z&x\end{vmatrix}$ last col 拆为 $\begin{pmatrix}0+y\\x-y+y\end{pmatrix}$, 按列 split $=\underline{\begin{vmatrix}x&\cdots&0\\&\ddots&\vdots\\&&x-y\end{vmatrix}+\begin{vmatrix}x&\cdots&y\\&\ddots&\vdots\\&&y\end{vmatrix}}=(x-y)\overbrace{\begin{vmatrix}x&\cdots&y\\\vdots&\ddots&\vdots\\z&\cdots&x\end{vmatrix}}^{n-1}+$ 各列上行 ×-1 加于此行 $\underline{\begin{vmatrix}x&y&\cdots&&y\\z-x&x-y&0&\cdots&0\\&&\ddots&&\vdots\\&&&x-y\\0&\cdots&0&z-x&0\end{vmatrix}}$

$=(x-y)D_{n-1}-(-1)^{1+n}\,y\,(z-x)^{n-1}$ expand last col

例6 求 k st. $\begin{vmatrix}b_1+c_1,&b_2+c_2,&b_3+c_3\\a_1+c_1,&a_2+c_2&a_3+c_3\\a_1+b_1,&a_2+b_2&a_3+b_3\end{vmatrix}=k\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}$

□ $=-\begin{vmatrix}-(b_1+c_1)&\cdots&-(b_3+c_3)\\&\cdots\end{vmatrix}=-\underbrace{\begin{vmatrix}2a_1&\cdots&2a_3\\&\cdots\end{vmatrix}}$ r2, r3 加于 r1 $=-2\underbrace{\begin{vmatrix}a_1&a_2&a_3\\c_1&c_2&c_3\\b_1&b_2&b_3\end{vmatrix}}=2\begin{vmatrix}a_1&\cdots\\b_1&\cdots\\c_1&\cdots\end{vmatrix}$ 提出 r1 的 2 再加于 r2, r3 消去 a

例7 111⋯1 □

determinent 1

det 为 minor

1. 令 $A_{n\times n}$, DEF $\widetilde{A}_{ij}$ 为 $(n-1)\times(n-1)$, 为 A 中去掉 i row j col.

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可 recursively def $\begin{cases} \det(A) = \sum_{j=1}^{n} \underbrace{(-1)^{1+j} A_{1j} \cdot \det(\widetilde{A}_{1j})}_{\text{cofactor } C_{ij}} \\ \text{若 } A_{2\times 2},\ \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc \end{cases}$, cofactor expansion (1st row)

cofactor C_{ij}, scalar, 称 A 的 adjugate 为 $adj(A)=C^T$

P_1 i) (det 为 linear function of row)

$$\det\begin{pmatrix} -a_1- \\ -u+kv- \\ -a_n- \end{pmatrix} = \underbrace{\det\begin{pmatrix} -a_1- \\ -u- \\ -a_n- \end{pmatrix}}_{B} + k\underbrace{\det\begin{pmatrix} -a_1- \\ -v- \\ -a_n- \end{pmatrix}}_{C} \leftarrow r \text{ row}$$

□ induction on n. 假设 n-1 成立, 对 n, $A_{r,:}$ 为 $(b_1+kc_1, \ldots) = \underset{(b_1\ldots)}{u} + k\underset{(c_1\ldots)}{v}$, 若 $r=1$, trival. 若 $r>1$, 考虑对各 j, $\widetilde{A}_{1j}$ 的 r-1 row 为 (删除1行, 往上移) $(b_1+kc_1, \ldots, b_{j-1}+kc_{j-1}, b_{j+1}+kc_{j+1}, \ldots)$, 即 $\widetilde{B}_{1j}$ 的 row r-1 + $\widetilde{C}_{1j}$ 的 r-1 row, 且其余 row $\widetilde{A}_{1j}$ 同 $\widetilde{B}_{1j}$ 同 $\widetilde{C}_{1j}$. 而这三个 n-1 方阵. ∴ 由 inductive hypothesis, $\det(\widetilde{A}_{1j}) = \det(\widetilde{B}_{1j}) + k\det(\widetilde{C}_{1j})$

$\det(A) = \sum_j (-1)^{1+j} A_{1j} \det(\widetilde{A}_{1j}) = \sum (-1)^{1+j} \underbrace{A_{1j}}_{r\neq 1,\ \because \text{1st row 不同}} \cdot \det(\widetilde{B}_{1j}) + k\sum_j (-1)^{1+j} A_{1j} \det(\widetilde{C}_{1j}) = \det B + k\det C$

ii) 一行全 0, det = 0 　　□ 由 i) $\det\begin{pmatrix} -a_1- \\ -z- \\ -a_n- \end{pmatrix} = \det\begin{pmatrix} -a_1- \\ -0\cdot z- \\ -a_n- \end{pmatrix} = 0\cdot\det\begin{pmatrix} -a_1- \\ -z- \\ -a_n- \end{pmatrix} = 0$ (z 为零 row)

iii) A row $i = e_k = (0 \cdots \overset{k}{1} \cdots 0)$, $\det A = (-1)^{i+k}\det(\widetilde{A}_{ik})$

□ 数归 n. 若 n-1 成立, 对 n, 若 $i=1$, 由 def 易证; 若 $i>1$, 考虑 $\forall j$, $\widetilde{A}_{1j}$ 的 i-1 row 为 $\begin{cases} e_{k-1} & j<k \\ 0 & j=k \\ e_k & j>k \end{cases}$ (shift 上 ↗ $\in F^{n-1}$; k-th 为 1; 1 to delete; k-th 为 1)

因 $\widetilde{A}_{1j}$ (n-1) 阶方阵, inductive hypothesis, $\det(\widetilde{A}_{1j}) = \begin{cases} (-1)^{i-1+k-1}\det(C_{ij}) & j<k \\ 0 \quad P_1\text{ ii)}\ j=k \text{ 行全 0} \\ (-1)^{i-1+k}\det(C_{ij}) & j>k \end{cases}$ ↕ 满足 induction 条件, row 有 e

C_{ij} 为 A 删去 row 1, i col j, k (对 j<k, $\widetilde{A}_{ij}$ 删 i-1 row k-1 col; j>k, $\widetilde{A}_{ij}$ 删 i-1 row k col)

∴ $\det A = \sum_j (-1)^{1+j} A_{1j} \det(\widetilde{A}_{1j}) = \sum_{j<k} (-1)^{1+j} A_{1j} (-1)^{i+k} \det(C_{ij}) + \sum_{j>k} (-1)^{1+j} (-1)^{i+k-1} A_{1j} \det(C_{ij})$

提取共同项 $= \underline{(-1)^{i+k}} \left[\sum_{j<k} (-1)^{1+j} A_{1j} \det(C_{ij}) + \sum_{j>k} (-1)^{1+j-1} A_{1j} \det C_{ij} \right]$ → 此为 $\widetilde{A}_{ik}$ 的 1st row cofactor expansion ∴ $=\det \widetilde{A}_{ik}$ (若为 1+(j-1) 因删了 k 列, shift 左, A_{1j} 变为 $(\widetilde{A}_{ik})_{1,j-1}$) 且因 k col 删, ∴ 无 gap

iv) (Laplace Expansion) cofactor expansion 可按 ∀ row 展开

□ 对 ∀i, WTS $\det A = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det \widetilde{A}_{ij}$. $i=1$ 即 def. $i>1$, row i 拆为 $A_{i1}\cdot e_1 + A_{i2}\cdot e_2 + \ldots + A_{in} e_n$ eg $(1\,3\,1) = 1\cdot(1\,0\,0) + 3\cdot(0\,1\,0) + 1\cdot(0\,0\,1)$

由 P_1 i), $\det A = A_{i1}\det\begin{pmatrix} -a_1- \\ -e_1- \\ -a_n- \end{pmatrix} + \ldots + A_{in}\det\begin{pmatrix} -a_1- \\ -e_n- \\ -a_n- \end{pmatrix} = \sum_j A_{ij} \det\begin{pmatrix} -a_1- \\ -e_j- \\ -a_n- \end{pmatrix} = \sum_j A_{ij} (-1)^{i+j} \det \widetilde{A}_{ij}$ 由 iii)

v) 二行相同, det = 0

□ induction n. 若 n-1 成立, 对 n, 令 row r, s 相同 选 i ≠ r≠s, $\det A = \sum_j (-1)^{i+j} A_{ij} \det \widetilde{A}_{ij}$ ↗ n-1×n-1, 且有相同行, ∴ det = 0

vi) ~~对 $\boxed{0}$ 的 row op,~~ 1) row scaling　$\det A \to \alpha\det A$

2) row exchange　$\det A \to -\det A$

3) row adjustment　$\det A \to \det A$

□ 1) 由 i)　2) 换 i, j row. $0 = \det\begin{pmatrix} -a_1- \\ -a_j+a_i- \\ -a_i+a_j- \\ -a_n- \end{pmatrix}$ (两行相同) $\xrightarrow{\text{apply i)}} = \det\begin{pmatrix} -a_1- \\ -a_j- \\ -a_i+a_j- \\ -a_n- \end{pmatrix} + \det\begin{pmatrix} -a_1- \\ -a_i- \\ -a_i+a_j- \\ -a_n- \end{pmatrix} \to \ldots = \underbrace{\det\begin{pmatrix} -a_1- \\ -a_j- \\ -a_i- \\ -a_n- \end{pmatrix}}_{\text{换后}} + \underbrace{\det\begin{pmatrix} -a_1- \\ -a_j- \\ -a_j- \\ -a_n- \end{pmatrix}}_{=0,\ v)} + \underbrace{\det\begin{pmatrix} -a_1- \\ -a_i- \\ -a_i- \\ -a_n- \end{pmatrix}}_{=0,\ v)} + \underbrace{\det\begin{pmatrix} -a_1- \\ -a_i- \\ -a_j- \\ -a_n- \end{pmatrix}}_{\text{原 matrix}}$

3) $\det\begin{pmatrix} -a_1- \\ -a_i- \\ -a_j+\alpha a_i- \\ -a_n- \end{pmatrix} = \underbrace{\det\begin{pmatrix} -a_1- \\ -a_i- \\ -a_j- \\ -a_n- \end{pmatrix}}_{\text{原 matrix}} + \alpha\begin{pmatrix} -a_1- \\ -a_i- \\ -a_i- \\ -a_n- \end{pmatrix}$ 两行相同, = 0

vii) 两行成比例, det = 0　□ 结合 i) 和 v)

viii) $\det(kA) = k^n \det(A)$　□ $\det\begin{pmatrix} -ka_1- \\ -ka_n- \end{pmatrix} = k\det\begin{pmatrix} -a_1- \\ -ka_2- \\ -ka_n- \end{pmatrix} = \ldots = k^n\det\begin{pmatrix} -a_1- \\ -a_n- \end{pmatrix}$

∴ 只当 n = even, $\det(-A) = \det A$　OR $\det kA = \det kI \cdot \det A = k^n \det A$

deli 得力

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例1 $A, B \in M_n(\mathbb{C})$, 若 B invertible, $\exists\, k \in \mathbb{C}$ st. $A+kB$ not invertible

□ $\det(A+kB) = \underline{\overset{\neq 0,\,去了}{\det B}} \cdot \underbrace{\det(B^{-1}A + kI_n)}_{可以看作\ B^{-1}A\ 的\ \text{characteristic polynomial}\ \boxed{25}.2}$ 由 fundamental thm of algebra $\exists$ zero(root) st. polynomial $= 0$

det 2

($\det A = 0$ 若可逆，有 $AB = I$，但 $\det I = 1 = \det A \cdot \det B = 0$)

, 交为 bicondition，$\det A \neq 0 \Leftrightarrow \operatorname{rnk} A = n$，可逆

不可逆，$\operatorname{rnk} A = \operatorname{rnk} L_A$，而 L_A bijection 为可逆

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1 $\mathcal{T}_1$ 若 $A_{n \times n}$，$\operatorname{rnk} A < n$，$\det A = 0$

☐ 23. P.22) rows L. dep. $\therefore$ 有 row $a_r = \sum_{i \neq r} k_i \cdot a_i$ 5 #2，$\therefore \det \begin{pmatrix} -a_1- \\ -a_r- \\ -a_n- \end{pmatrix} = \det \begin{pmatrix} -a_1- \\ -k_1a_1 + \dots + k_na_n- \\ -a_n- \end{pmatrix} = \det \begin{pmatrix} -a_1- \\ -0- \\ -a_n- \end{pmatrix} = 0$

row op type 3, add $-k_i \cdot a_i$ 在 a_r, P5. P1 vi)3)

行 = 0, det = 0

$\mathcal{T}_2$ 对 $A, B \in M_n(\mathbb{F})$，$\det AB = \det A \cdot \det B$

☐ lemma: 对 elementary E，$\det EB = \det E \cdot \det B$. 四. $\mathcal{T}_1$

△ 若 E type 1)，LHS = 等效于 B row scale，$= \alpha \cdot \det B$

RHS $= \underset{=}{\alpha} \cdot \det B$ …

$\det(\,{}^{1}\alpha_{\cdot 1}) = \alpha$

当 $\operatorname{rnk} A = n$. A 可逆 $\therefore A = E_m \cdots E_1$ 23. P.iii

$\therefore \det AB = \det(E_m \cdots E_1 B) \overset{\text{lemma}}{=} \det(E_m) \cdot \det(E_{m-1} \cdots E_1 B)$

$= \dots = \det(E_m) \cdots \det(E_1) \det B = \det(E_m \cdots E_1) \det B$

lemma 反过来看，聚合

$= \det A \cdot \det B$

若 $\operatorname{rnk} A < n$，$\overset{\mathcal{T}_1}{\det A = 0}$，$\therefore$ RHS = 0. 而对 AB，$\operatorname{rnk} AB \leq \operatorname{rnk} A < n$，$\therefore$ LHS = 0 23. $\mathcal{T}_2$ c)

$\mathcal{T}_3$ $\det A^T = \det A$

☐ $\operatorname{rnk} A < n$，但 $\operatorname{rnk} A^T = \operatorname{rnk} A$，$\therefore$ det 都为 0. ; A invertible，$A = E_m \cdots E_1$，而 $\det E_i = \det E_i^T$

因 (对各 type I & II symmetry)

$\therefore$ 有 $\det A^T = \det(E_1^T \cdots E_m^T) = \det(E_1^T) \overset{\mathcal{T}_2}{\cdots} \det(E_m^T) = \det(E_m) \cdots \det(E_1) = \det(E_m \cdots E_1) = \det A$

四. 例1

因 这表明 Laplace expansion 也可沿 ∀ col 展开，增强 P5. P1 iv)

$\mathcal{T}_4$ $\det A = \prod_{i=1}^{n} \lambda_i$ λ_i: eig val

有 n root

☐ characteristic func $f(\lambda) = (-1)^n (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) = -(\lambda_1 - \lambda) \times -(\lambda_2 - \lambda) \times \cdots \times -(\lambda_n - \lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$

因 λ 为变量，令 $\lambda = 0$ 则 $f(0) = \prod_{i=1}^{n} \lambda_i = \det(A - 0 \cdot 1)$ NOTE，即 5 $\mathcal{T}_{10}$ 的 NOTE，$k_0 = \prod^{n} \lambda_i$

OR 若可 eig decomposition $A = Q \Lambda Q^{-1}$，$\det A \overset{\mathcal{T}_2}{=} \det Q \det \Lambda \det Q^{-1} = \det \Lambda = \prod^{n} \lambda_i$

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Vector Space

n维向量空间 [illegible] n-dim 向量空间

let K field, (Vector Space 特例, $K^n = \{(a_1; \dots ; a_n) \mid a_i \in K\}$)

V over K satisfy 以下 即 V vector space

$+ : V \times V \to V$, $\cdot : K \times V \to V$ (operation 本包括 closure)

(V,+) abelian group

(A1) $\forall x, y \in V$, $x+y=y+x$ commutative

(A2) $\forall x,y,z \in V$, $(x+y)+z = x+(y+z)$ associative

(A3) $\exists!\ \underline{0} \in V\ \forall x \in V$, $x+0=x$ identity ($(0; \dots ; 0) = \vec{0}$)

(A4) $\forall x \in V, \exists!\ y \in V$, $x+y=0$ inverse (let $x=(x_1; \dots ; x_n) \in K^n$, $y=-x=(-x_1; \dots ; -x_n)$)

monoid

(M1) $\forall a,b \in K, \forall x \in V$, $(ab)x = a(bx)$ associative

(M2) $\exists!\ \underline{1} \in K, \forall x \in V$, $1 \cdot x = x$ identity

(D1) $\forall x,y \in V, \forall c \in K$, $c(x+y) = cx+cy$

(D2) $\forall x \in V, \forall c,d \in K$, $(c+d)x = cx+dx$

! 不代表 $ax \neq bx \Rightarrow a=b$ $(x=0)$ 或 $ax=ay \Rightarrow x=y$ $(a=0)$

1. i) cancel law: $\forall x,y,z \in V$, $x+z=y+z \Rightarrow x=y$

ii) A3, A4, M2 都 unique   如A3 $0+0'=0$ ($0'$ unique) $\therefore 0=0'$

$0+0'=0'$ (0 unique)

iii) ∈K, identity for addition

$\underline{0}x = \underset{\in V}{\underline{0}}\ \forall x \in V$. $0x = \underbrace{(0+0)x = 0x+0x}_{D_2}$, $\therefore 0x=0$ (1. i) cancel law)

iv) $x\underset{\in V}{0} = \underset{\in V}{\underline{0}}\ \forall x \in K$ $x0 = \underbrace{x(0+0) = x0+x0}_{D_1}$

任意 set!!

2. $\phi \subset \underline{\underline{S}}$, F field, $\mathcal{F}(S,F)$ set of all $f: S \to F$. let $f,g \in \mathcal{F}(S,F)$, $c \in F$

$f=g$ iff $f(s)=g(s)\ \forall s \in S$, $+ : (f+g)(s) = f(s)+g(s)$, $\cdot : (cf)(s) = cf(s)$, 0为 $f(x)=0$

3. F field, $\mathbb{P}[F]$ 为 coeff ∈ F 的 polynomial space. $P_n(x) = a_0 + \dots + a_n x^n \in \mathbb{P}[F]$, degree n

degree为n不是! $x^2-x^2=0$, 非degree 2   0为 $0x^0 = 0 \in \mathbb{P}[F]$

事实上 $\mathbb{P}_n[F]$ 为 $\mathbb{P}[F]$ 中 <u>degree at most</u> n, 也为 vector space over F. ($\mathbb{P}_n(F)$ 为 subspace of $\mathbb{P}(F)$)

且 $P_n(x) = a_0 + \dots + a_n x^n \in \mathbb{P}_n[F]$ 可写为 $\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$

4. • 最trivial的 vec. space over any field F is $\{0\}$, $0+0=0$, $a0=0$

• (seq space) Seq(F) 为 ele in F 的 seq. $\{s_n\}+\{t_n\}=\{s_n+t_n\}$, $c\{s_n\}=\{c \cdot s_n\}$

• (coordinate space) $(a_1, \dots a_n)$, $a_i \in F$: F^n

• (Complex) $\mathbb{C}$ over $\mathbb{C}$ basis $\{1\}$, $\mathbb{C}$ over $\mathbb{R}$ basis $\{1, i\}$, 但 $\mathbb{R}$ over $\mathbb{C}$ 非 VS. $i \cdot 1 = i \notin \mathbb{R}$

vs2

Date. / 7

1 $\mathbb{R}^+$ all positive reals. $\mathbb{R}^+$ over $\mathbb{R}$ VS, addition : $x \boxplus y = xy$
scalar multi : $\alpha \odot y = y^\alpha$

dim = 1, basis = $\{10\}$ ($\forall x \in \mathbb{R}^+$, $x = (\log_{10} x) \odot 10$ 且 = ele ~ 2 L.dep)

例1 判断 : 1° $V = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{F}\}$, +: coord 加法, ·: $c(a_1, a_2) = (a_1, 0)$

2° $V = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$, + $(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$, · $c(a_1, a_2) = (ca_1, ca_2)$

答 1° ✗ ∵ 因 $0 \cdot (a_1, a_2) = \underset{\sim}{(a_1, 0)}$ zero vec. 不 unique 2° ✗ $[(a_1, a_2) + (b_1, b_2)] + (c_1, c_2) \neq (a_1, a_2) + [(b_1, b_2) + (c_1, c_2)]$

subspace 不一定将 x+y 包括

1. subspace $W \subseteq$ vector space V

应满足 vector space 的性质. P8 所8点 只有(A3), (A4), 和 "+"/"·" 的 closure 不可推证 subspace 同性质 （系数本）

而 回.5 的 3条 ("+"/"·" closure + (A3)) 可证 (A4)

☐ $0 \in W$ ×
☐ $\forall a \in F, x,y \in W$
$ax+y \in W$ 也可证 W 为 V subspace

☐ let x inverse 为 $(\underline{-1})\cdot x$, $x+(-1)\cdot x \overset{M2;\ x=1\cdot x\ \&\ D2}{=} (1+(-1))\cdot x \overset{P8.1.iii)}{=} 0\cdot x = 0$

feid multi identity 的 add inverse

NOTE, 由 $\forall x \in V, 0\cdot x = 0$ (P8. 1. iii)) 可知: $0 \in V$ 总在任意 V 的满足 "·" closure 的 subset

但 subspace 的 "zero vector in W" 还保障了至少 W 非空 (对 ϕ, closure trivilly true)

eg 1. W 为 set of symmetric matrix in $M_{n\times n}(F)$, W 为 $M_{n\times n}(F)$ 的 subspace. ☐ $(A+B)^T = A^T+B^T = A+B \in W$

2. $F(R,R)$ 的 subspace 为 all continuous func: $R\to R$ ☐ sum of cont. ∈ cont.

3. set of diagonal matrix ($M_{ij}=0$ 当 $i\neq j$) subspace of $M_n(F)$ / set of upper triangular matrix subspace of $M_n(F)$ set of $M_{n\times n}(F)$ trace $=0$

对 partial order with $\subseteq$, $\cap C_i$ 即 meet, $\wedge C_i$

2. P_1 intersection of subspace of VS. V 也为 V 的 subspace 因 intersection 是 largest subset of V

☐ let C 为 V 的所有 subspace, W 为 C 的 ele 的 intersection ($\cap C_i$). $0 \in W$
$x+y, cx \in W$ (x, y 在所有 C_i 中)

NOTE. union 非 subspace, 如 $V=\mathbb{R}^2$, $W_1 = \{\begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R}\}$, $W_2 = \{\begin{bmatrix} 0 \\ x \end{bmatrix} : x \in \mathbb{R}\}$, 见 P_2

例1 V 为 V.S. 且 $W \subseteq V$ 也为 V.S., W 为 V subspace × 除非 "+", "·" 也完全一致 如 $V=\mathbb{R}$ on field $\mathbb{R}$ 但 $W=\mathbb{Q}$ 为 V.S. on $\mathbb{Q}$ 而 非 $\mathbb{R}$

W 为 $\mathbb{R}^3$ 的 xy-plane, 即 $W=\{(a_1,a_2,0)\}$, 则 $W=\mathbb{R}^2$ × $W \cong \mathbb{R}^2$ (isomorphism), $w \in W$ 有 3 coord, 但 $x \in \mathbb{R}^2$ 只有 2 个

P_2. W_1, W_2 为 V 的 subspace, $W_1 \cup W_2$ 为 V subspace $\Leftrightarrow W_1 \subseteq W_2 \vee W_2 \subseteq W_1$

☐ $\Leftarrow$ trival, $\Rightarrow$, 若两者都不对, 则 $\exists x \in W_1 - W_2, y \in W_2 - W_1$. 则 $x,y \in W_1 \cup W_2$, $x+y$ 也 $\in W_1 \cup W_2$ 那么若 $x+y \in W_1$, $x+y+(-x) = y \in W_1$ 矛盾 $x+y \in W_2$ 同理

对 $W_1, \dots, W_n \subseteq V$, 若 $\cup W_i \subseteq V$ $(\subseteq V)$ $\Leftrightarrow$ 有 W_k contain all. 见右.5.

3. 对 W_1, W_2, 令 $W_1+W_2 = \{(x+y) \mid x \in W_1, y \in W_2\}$, if $W_1 \cap W_2 = \{0\} \wedge W_1+W_2 = V$, 则 $V = W_1 \oplus W_2$

(V subspace) 即 $w_1+w_2=0 \Rightarrow w_1=w_2=0$

T_1 若 W_1, W_2 为 V 的 subspace, W_1+W_2 也为 subspace; any subspace of V contains both W_1, W_2 must also contains W_1+W_2

☐ i) $0=0+0 \in W_1+W_2$
且: $x_1+y_1+x_2+y_2 \in W_1+W_2$
ii) 因 W subspace, 令 $x \in W_1$, $y \in W_2$ $x+y$ 必在 W. $\therefore W_1+W_2 \subseteq W$

例2. 证 $\mathbb{F}^n = \{(a_1, \dots, a_n) \mid a_n = 0\} \oplus \{(a_1, \dots, a_n) \mid a_1 = \dots = a_{n-1} = 0\}$

☐ $W_1 \cap W_2 = \{0\}$, 而 $\begin{cases} W_1+W_2 \subseteq V: & \text{易证 } W_1 W_2 \text{ subspace of } \mathbb{F}^n. \therefore \underline{W_1+W_2 \text{ 也为 subspace}} \ (T_1). \text{ subspace 本为 subset} \\ V \subseteq W_1+W_2: & \text{let } v=(v_1,\dots,v_n) \in V, \ v=(v_1,\dots,v_{n-1},0)+(0,\dots,0,v_n) \text{ 为 subspace, 只提} \end{cases}$

例3 证当 F 非 characteristic 2, $M_n(F) = \{\text{set of skew-symmetric}\} \oplus \{\text{set of symmetric}\}$ (eg 1)

☐ $W_1 \cap W_2 = \{0\}$, 而 $M_n(F) = \{A : A \in M_n(F)\} = \{(\underline{A+A^T}) + (\underline{A-A^T}) \mid A \in M_n(F)\} = W_1+W_2$

若 F characteristic 2, $F \cong \mathbb{Z}_2$ $M^T = M = -M^T$ $\begin{pmatrix} -1=1 \\ -0=0 \end{pmatrix}$ 因 页首, Left symmetric, Right skew symmetric

事实上, F 非 characteristic 2 时, $M_n(F) = \{\text{set of symmetry}\} \oplus \{\text{set of } A,\ A_{ij}=0\ i \leq j\}$

☐ $W_1 \cap W_2 = \{0\}$, 而 $M_n(F) = \{A : A \in M_n(F)\} = \{T + A - T\} = W_1+W_2$. $T_{ij} = T_{ji} = A_{ij}$ 选择 $T \in W_1$ 且 $A-T$ 的上三角 $=0$, $\in W_2$

NOTE, 由例3, 3. 的 $W_1 W_2$ 不一定 unique:

T_2 W_1, W_2 subspace of V, $W_1 \oplus W_2 = V \Leftrightarrow \forall v \in V$, $v = x_1+x_2$ uniquely, $x_1 \in W_1$, $x_2 \in W_2$

☐ $\Rightarrow$ 若不同 present, $v = x_1+x_2 = x_1'+x_2'$ $\underset{\in W_1}{\underline{x_1'-x_1}} = \underset{\in W_2}{\underline{x_2-x_2'}}$ 但 $W_1 \cap W_2 = \{0\}$ $\therefore x_1 = x_1'$, $x_2 = x_2'$

$\Leftarrow$ 若 $W_1 \cap W_2 \neq \{0\}$, $\exists t \in W_1 \cap W_2$ 而 $t \neq 0$, 则 $t = t+0 = 0+t$, 不 unique

VS4

4. eg1. 对 vec spa V over F, 令 $v \in V$, $\langle v \rangle = \{av \mid a \in F\} \leq V$, 为包含 v 的最小 subspace

类似 group theory 的 generator, 但不像 coset, aH 的 a 为 G 的 ele, 而此处 $a \in F \notin V$

事实上 $\langle v \rangle = \text{span}(\{v\})$ 见 $P_2 T_1$), $\langle v \rangle \leq V$

eg2. $W \leq V$, $v+W = \{v+w \mid w \in W\}$ 为 coset. (L. Alg 称为 affine subspace)

类似 group theory. $v+W \leq V \Leftrightarrow v \in W$; $v_1+W = v_2+W \Leftrightarrow v_1 - v_2 \in W$

□ i) $\Rightarrow$ $0 \in v+W$, $v+?=0$, $?=-v \in W$, $\therefore v \in W$. $\Leftarrow$ W closed

ii) $\Rightarrow$ $v_1+w = v_2+w'$, $v_1 - v_2 = \underline{W-w} \in W$. $\Leftarrow$ $\forall e = v_1+w \in v_1+W$, since $t = v_1 - v_2 \in W$, $v_1 = t+v_2$, $e = v_2+t+w \in v_2+W$

而 $v_2+W \leq v_1+W$ 同理

定义 coset 的两 operation: $(v_1+W)+(v_2+W) = (v_1+v_2)+W$

$a(v_1+W) = (av_1)+W$

可见 quotient space V/W 为 V.S.

5. P_{10} P_2 的脚注: $\Rightarrow$. 反证, 若无 subspace contains all. 则 $\exists u \in S_1 \setminus (S_2 \cup .. \cup S_n)$

且 $\exists v \in (S_2 \cup .. \cup S_n) \setminus S_1$ 若 $au+v \in S_1$, 因 $u \in S_1$, $-au \in S_1$, $au+v-au = v \in S_1$, 矛盾.

若 $au+v \in S_k$ for some k, 则又有 $b \neq a$, $bu+v \in S_k$, 则 $(a-b)u \in S_k$ 但 $u \in S_1$ 矛盾.

若拓展至 infinite. 对 partial ordered set $(P(V), \subseteq)$ 的特例 $(\text{sub}(V), \subseteq)$, 其 infinite chain

$C = \{W_1, ..\}$ st. $W_1 \subseteq W_2 \subseteq ..$, $\overset{\text{union}}{\underline{\cup}} C$ 为 subspace of V. P_2

□ 若 $u, v \in \cup C$, $u \in W_i$, $v \in W_j$, 令 $j > i$, 则 $\underline{u+v \in W_j}$

6. 用 direct product 增加维度, 对 $V_1, .., V_n$ over $\underline{\text{same}}$ field F,

$\prod^{n} V_i = \{(v_1, .., v_n) \mid v_i \in V_i\}$ 为 V.S. over F,

addition: $(v_1, .., v_n)+(u_1, .., u_n) = (v_1+u_1, .., v_n+u_n)$

multiplication: 才能 st. $\alpha v_i \in V_i$ $\alpha(v_1, .., v_n) = (\alpha v_1, .., \alpha v_n)$

eg. $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ over $\mathbb{R}$

7 T_3 对 V.S. V, $W < V$, $\exists T < V$ st. $W \oplus T = V$

□ let F family of all sub(V) st. disjoint from W. 对 order operation $\subseteq$, let C 为 F 的 chain,

据5. $\underline{\cup C \leq V}$, 且 $\cup C \cap W = \{0\}$, 因 V 的 upper bound, 由 Zorn's lemma, $\exists$ maximal of F: M

if $W \oplus M < V$, 则 $\exists e \in V-(W \oplus M)$, $e \neq 0$, ???

(可由 P_{12} T_{20} 证明)

NOTE, 学习了 basis 后, 可用 basis 证明: 易证 当 $W, T \leq V$, $\beta_W \cap \beta_T = \{0\}$, $V = W \oplus T \Leftrightarrow \beta_W \cup \beta_T = \beta_V$

则对 W 的 basis β_W, def $T = \text{span}(\beta_V - \beta_W)$ 由 lemma, $W \oplus T = V$

In fact T 不唯一, 如 $\mathbb{R}^2 = V$, $W = \{(a_1, 0) \mid a_1 \in \mathbb{R}\}$, T 可取 $\{(0, a_2) \mid a_2 \in \mathbb{R}\}$ 也可取 $\{(a_2, a_2) \mid a_2 \in \mathbb{R}\}$

例1 $W_1 = \{\text{set of odd func}\}$, $W_2 = \{\text{set of even}\}$ $\overset{\text{连续函数}}{C(\mathbb{R})} = W_1 \oplus W_2$

□ 对 $\forall f \in C(\mathbb{R})$ $f(x) = \underbrace{\frac{1}{2}(f(x) - f(-x))}_{\text{odd, } g(-x) = -g(x)} + \underbrace{\frac{1}{2}(f(x) + f(-x))}_{\text{even, } g(-x) = g(x)}$

同见 ***10　　dim

例1. 对 P10. eg1.3，求 dim. & symmetric & ~~skew~~-symmetric

☐ diagonal: $\{E_{ii}\}$. dim $= n$， trace $=0$：可为 $\begin{bmatrix} -a_{22}-a_{33}\cdots-a_{nn} & & \\ & a_{22} & \\ & & \ddots \\ & & & a_{nn}\end{bmatrix} = \{E_{ij} \mid i\ne j\} \cup \{\begin{bmatrix}-1 & \\ & 1 \\ & & \ddots & 0\end{bmatrix}, \begin{bmatrix}-1 & \\ & 0 & \\ & & 1 & \\ & & & \ddots & 0\end{bmatrix}, \ldots, \begin{bmatrix}-1 & \\ & \ddots & \\ & & 1\end{bmatrix}\}$

同 $\{E_{ii}-E_{11} \mid 2\le i\le n\}$

upper triangle: $\{E_{ij} \mid i\le j\}$ dim $= \frac{(1+n)n}{2}$　　dim $= n^2-1$，tr 可视为 L.T. $M_{n\times n}(F)\to F$，nullity $= n^2-1$，rank $= 1$，且 1-1，onto

symmetric：同 upper triangle　　skew-symmetric $\begin{bmatrix} 0 & \cdots & ? \\ -? & & 0\end{bmatrix}$，$\{\begin{bmatrix} 0 & 1 \\ -1 & \ddots & 0\end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & \ddots \\ -1 & & 0\end{bmatrix}, \ldots, \begin{bmatrix} \ddots & \\ & -1 & 0\end{bmatrix}\} = \{E_{ij}-E_{ji} \mid i<j\}$

dim $= \frac{(1+n-1)(n-1)}{2} = \frac{n^2-n}{2}$

1 T_1 $W_1, W_2 \le V$, dimV finite. $W_1 \subseteq W_2 \Leftrightarrow \dim(W_1\cap W_2) = \dim(W_1)$

☐ $\Rightarrow W_1\cap W_2 = W_1$. $\Leftarrow$ let $\dim W_1 = n$，β 为 basis for $W_1\cap W_2$. note $W_1\cap W_2 \subseteq W_1$，

$\beta\subseteq W_1$，且因 $\dim W_1 = n$，β 本为 n 个 L.Ind，$\therefore$ β 为 W_1 basis ***3 P_8 同 basis，这两空间相同，$W_1\cap W_2 = W_1$ $\therefore W_1 \subseteq W_2$

OR: $W_1\cap W_2 \le W_1$ 且两者 dim 相同，$W_1\cap W_2 = W_1$ (***3 T_3 来答) (W_1 本为 subspace cp-vector space，$\therefore W_1 \le W_2$)

T_{2a} $W_1, W_2\le V$ 且 finite dim，和 sup space W_1+W_2 有 $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1\cap W_2)$

☐ let $\{u_1, \ldots, u_k\}$ basis for $W_1\cap W_2$，since $W_1\cap W_2 \subseteq W_1, W_2$，extend to (***3 $P_{(.)}$) $\{u_1, \ldots, u_k, v_1, \ldots, v_m\}$ basis for W_1

$\{u_1, \ldots, u_k, w_1, \ldots, w_n\}$ basis for W_2

WTS $\{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n\}$ basis for W_1+W_2　LHS表示 $\in W_1$ (basis为$\{u_i, v_i\}$)，RHS表示 $\in W_2$，$\therefore$ 同时在 both

i) L.Ind. let $a_i, b_i, c_i \in F$，$\underline{\sum a_i u_i + \sum b_i v_i = -\sum c_i w_i \in W_1\cap W_2}$

$\therefore -\sum c_i w_i$ 可由 $\{u_i\}$ 表示：$-c_1w_1 - \cdots - c_nw_n = t_1u_1 + \cdots + t_ku_k$，这为 W_2 basis 的 L.Comb.　L.Ind. $c_i = t_i = 0$

同理 $\sum a_iu_i + \sum c_iw_i = -\sum b_iv_i$，$b_i = 0$，$\therefore \sum a_iu_i + 0 + 0 = 0$，由 $W_1\cap W_2$ basis，$a_i = 0$

ii) span： $x\in W_1+W_2$ 写为 $(\sum a_iu_i + \sum b_iv_i) + (\sum c_iu_i + \sum d_iw_i)$

T_{2b} $V = W_1+W_2$，则 $V = W_1\oplus W_2 \Leftrightarrow \dim V = \dim W_1 + \dim W_2$

☐ $W_1\cap W_2 = \phi \Leftrightarrow \dim W_1\cap W_2 = 0 \Leftrightarrow \dim V = \dim W_1 + \dim W_2$ (T_{2a})

例2. $W_1, W_2 \le V$，$\dim(W_1\cap W_2) \le \min(\dim W_1, \dim W_2)$

$\dim(W_1+W_2) \le \dim(W_1) + \dim(W_2)$ $\nearrow$ ***3 T_3　　非负

☐ let $\dim W_1 \le \dim W_2$，note $\underline{W_1\cap W_2 \le W_1}$ ($W_1\cap W_2$ v.s. (P_{10} $P_{?}$) 且 subset of W_1) $\therefore$ $\underline{\dim LHS \le \dim W_1}$；$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \underline{\dim(W_1\cap W_2)}$

Note：例2 i) 等号：$W_1 = \{\begin{bmatrix} a \\ b \\ 0\end{bmatrix}\}$ $W_2 = \{\begin{bmatrix} a \\ 0 \\ 0\end{bmatrix}\}$ $W_1\cap W_2 = \{\begin{bmatrix} a \\ 0 \\ 0\end{bmatrix}\}$ dim $= 1$

ii) 等号 $W_1 = \{\begin{bmatrix} a \\ b \\ 0\end{bmatrix}\}$ $W_2 = \{\begin{bmatrix} 0 \\ 0 \\ c\end{bmatrix}\}$ $W_1+W_2 = \{\begin{bmatrix} a \\ b \\ c\end{bmatrix}\}$ dim $= 3$

2 T_{2c} $\dim(V) = \dim(W) + \dim(V/W)$　P114.

☐ 令 $\beta_W = \{v_1 \cdots v_k\}$，$\beta_V = \{v_1, \cdots v_n\}$ (extend by β_W ***3. $P_{(.)}$)，WTS $\{v_{k+1}+W, \ldots, v_n+W\}$ basis for V/W

i) L.Ind $\sum_{i=k+1}^{n} k_i(v_i+W) = \underbrace{\sum_{i=k+1}^{n} (k_iv_i+W)}_{\text{scalar multiplication def}} = \underbrace{\sum (k_iv_i)+W}_{\text{addition def}} = \underbrace{0+W}_{\text{zero vector}}$ $\therefore \sum_{i=k+1}^{n} k_iv_i \in W$ (P114 性质)

但这部分 v_i 不在 W 中，$\therefore \sum k_iv_i$ 只能 $= 0$，而 v_i L.Ind. $\therefore k_i = 0$

ii) span. $\forall x+W \in V/W$，$x+W = (\sum^{k}_{i} a_iv_i + \sum^{n}_{j} a_jv_j) + W$. 因 $\sum^{k} a_iv_i \in W$，$\therefore = \underline{\sum a_jv_j + W}$ (P114 性质)

NOTE，以 L.T. 的角度证明：let $\eta: V\to V/W$，$v\to v+W$，易证 η linear

$\text{Null}(\eta) = \{v\in V \mid v+W = 0+W\}$ that is，$v\in W$，$\therefore \text{Nul}(\eta) = W$，

$\therefore$ 由 rank Thm，$\dim V = \text{rank}\,\eta + \text{nullity}\,\eta = \dim V/W + \dim W$　由 def 得

3. 有 $T: V\to Z$ linear 且 onto，def $\overline{T}: V/\text{Nul}(T) \to Z$，$v+\text{Nul}(T) \to T(v)$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rd, "\eta"'] & Z \\ & V/\text{Nul}(T) \arrow[u, "\overline{T}"'] \end{tikzcd}$$

证 $\overline{T}$ well-defined 且为 isomorphism，且 $T = \overline{T}\eta$

☐ well-defined：$v+N(T) = v'+N(T) \Rightarrow \underline{T(v) = T(v')}$ $\overline{T}$ map 到同一处　(因 $\underline{v-v' \in \text{Nu}(T)}$，$T(v-v') = 0$，$T(v) - T(v') = 0$) P114 性质

linear 易证，onto：因 T onto 对 $z\in Z$，有 $T(x) = z$ $\therefore$ 有 $\overline{T}(x+\text{Nu}(T)) = T(x) = z$

1-1：令 $\overline{T}(x+N(T)) = T(x) = 0$，$\underline{x\in \text{Nul}(T)}^{\text{def}}$ $\therefore$ $\underline{x+\text{Nu}(T) = \text{Nul}(T)} = \overline{0}$ coset zero vector (P114 性质)

对 $\forall x\in V$ $T(x) = \overline{T}(x+\text{Nul}(T)) = \overline{T}\eta(x)$

例1. $V_1, V_2 \leq V$, 若 $\dim(V_1+V_2) = \dim(V_1 \cap V_2) + 1$, 证 $V_1+V_2 = V_i$ ($V_1 \cdot V_2$中一个)

$V_1 \cap V_2 = V_{1-i}$ (另一个)

□ 由 $V_1 \cap V_2 \leq V_i \leq V_1+V_2$ $\dim(V_1 \cap V_2) \leq \dim V_i \leq \dim(V_1+V_2)$ ~~xxx~~ 3. J_3

∴ 有 要么 $\dim V_i = \dim(V_1 \cap V_2)$ or $\dim(V_i) = \dim(V_1+V_2)$ → 同理 ∴ ~~两种情况都成立~~

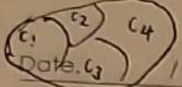
→ 则 $V_{1-i} \supseteq V_i$, 则 $V_{1-i} = V_1+V_2$

R_2 J_1

$\dim(V_1 \times V_2) = \dim(V_1) + \dim(V_2)$

$\dim(V_1 \otimes V_2) = \dim(V_1) \times \dim(V_2)$

1. 对 ~ equivalent relation on S. 因 equivalent relation 可记为 partition (Algebra P17. 2 Lemma)



为了快速了解 $\forall s \in S$, s 属于哪个 equivalent class, 在各 equivalent class E_i 中选一个 c_i, 将 s 与 c_i 比较:

$C \subseteq S$ 为 system of distinct representatives for ~ on S iff $\forall E_i$ contains 1 member in C

C 也称为 set of canonical form

eg1. let $A, B \in M_{m,n}(F)$, $\underset{\text{称A,B equivalent, 事实上为 equivalent relation}}{}\overline{A \equiv B}$ iff $\exists$ invertible $P \in M_m(F)$, $Q \in M_n(F)$, $B = PAQ$

即: A 可由 row/col operations reduce 为 B

事实上, $\forall$matrix R 会被 reduce 为 $J_r = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$.

且 no distinct $J_s \equiv J_r$ □ if $J_s = PJ_rQ$, $\mathrm{rank}(J_s) = s = \mathrm{rank}(PJ_rQ) = \mathrm{rank}(J_r) = r$

这样 $C = \{J_r \mid r \geq 0\}$ 为 set of canonical form, $A \equiv B$ iff A, B 有同 canonical form J_s

2. 用 func 了解 equivalent class

constant on equivalent class

$f: S \to X$ 为 invariant under ~ (on S) iff $a \sim b \Rightarrow f(a) = f(b)$

complete invariant under ~ iff $a \sim b \Leftrightarrow f(a) = f(b)$

eg2. 对 equivalent relation $\equiv$ (见eg1), rank 为 complete invariant func

为eg1. 的 $\equiv$

□ $A \equiv B \Leftrightarrow \mathrm{rank}(A) = \mathrm{rank}(B)$

row operation

eg3. $A, B \in M_n(F)$, A B similar ($A \sim B$) iff $\exists$ invertible P, $B = P^{-1}AP$

可证 $A \sim B$ 为 equivalent relation on $M_n(F)$

Jordan

rational canonical form

matrix similar 有多种 invariants 但无 complete invariants

如 det: □ $\det(B) = \det(P^{-1})\det(A)\det(P) = \det(A)$, $\therefore$ invariant, 但不同 matrix 会有同det

trace: □ $\mathrm{tr}(B) = \mathrm{tr}(P^{-1}AP) = \mathrm{tr}(AP^{-1}P) = \mathrm{tr}(A)$, 但 $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ trace同 $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ 但非 similar

NOTE: complete invariant 可完全决定 ele 属于哪个 equivalent class, 只用施加 func

但 complete invariant 很少, 一般用 noncomplete 的选否来判断 ele 不属于哪个 class

Inverse 4 ← $\boxed{15}$

Date. /

此时 inv $= A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}$

证 1. (Matrix Inverse Lemma) 对可逆A，$A+uv^T$ 可逆 ⟺ $v^TA^{-1}u \neq -1$.

$A+uv^T$ & A^{-1} 可逆，若$=0$，可逆 $u=0=u$

□ ⟹ $[(A+uv^T)A^{-1}]u \neq 0 \Leftrightarrow u+uv^TA^{-1}u \neq 0 \Leftrightarrow u(1+v^TA^{-1}u) \neq 0$

⟸ Note $P = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}$ 是 $Q = A+uv^T$ 的逆. By check $PQ=QP=1$

注 $A=C=I$ reduce为证4. lemma 2，并从lemma 2 令 $U=A^{-1}X$, $V=CY$ 还原

证 2 (Sherman-Morrison-Woodbury) A_{nn} 可逆，有对应的 U_{nk} V_{kn} C_{kk}

代数证 = I $A+UCV^T$ 可逆 ⟺ k×k 的 $(C^{-1}+V^TA^{-1}U)$ 可逆，此时 $(A+UCV^T)^{-1} = A^{-1} - A^{-1}U(C^{-1}+V^TA^{-1}U)^{-1}V^TA^{-1}$

—

证 3 $(I+P)^{-1} = I-(I+P)^{-1}P$ □ 令 $U \leftarrow I+P$ $U(I-U^{-1}P) = U-P = I$ 证4. lemma 2 令 $A \leftarrow I$, $B \leftarrow P$

证4 (Push-Through) $U_{nd}^T(I_n+V_{nd}U^T)^{-1} = (I_d+U^TV)^{-1}U^T$

(lemma 1) $A(I+BA)^{-1} = (I+AB)^{-1}A$ $A(I+BA) = (I+AB)A \Leftrightarrow A(I+BA)^{-1} = (I+AB)^{-1}A$

(lemma 2) $(I+AB)^{-1} = I - A(I+BA)^{-1}B$ 考虑 $(\underline{I-(I+AB)^{-1}}AB)(I+AB) = I+AB-(I+AB)^{-1}AB(I+AB)$

$= I+AB-(I+AB)^{-1}(I+AB)(AB) = I+AB-AB = I$, 即 $I - A(I+BA)^{-1}B$, lemma 1

∴ 有 lemma 1 → Push-Thru

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例 1　inv A　10×10 A $\begin{pmatrix} 2 & & 1 \\ & 2 & \\ 1 & & 2 \end{pmatrix}$　口 $A = I + uu^T$, $u = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ 逆 $= I - \dfrac{uu^T}{1+10} = \begin{cases} 10/11 & \text{diag} \\ -1/11 & \end{cases}$

norm 3 ← P52

Date. /

1 Frobenius/energy $\|A\|_F \underset{B.1.}{=} \|A^T\|_F = \sqrt{\sum_{ij} a_{ij}^2} = \|vec(A)\|_{\ell_2}$

note $\|A\|_F^2 = tr(AA^T) = tr(A^TA)$ ☐ $(A^TA)_{ij} = \sum_k A^T_{ik}A_{kj} = \sum_k A_{ki}A_{kj}$

P1a) 为 matrix norm

☐ i ii) trivl (iii) 令$\hat{A}\hat{B}$=vec(A), vec(B) 由vec norm 的iii) (iv) $\|AB\|_F^2 = \sum_{ij}(\bar{a}_i b_j)^2 \overset{\text{Cauchy P52, P1)}}{\le} \sum_{ij}\|\bar{a}_i\|^2\|b_j\|^2$

$= (\sum_i\|\bar{a}_i\|^2)(\sum_j\|b_j\|)^2 = \|A\|_F^2\|B\|_F^2$

1b) 对 vec $\vec{x}\,\vec{y} \in \mathbb{R}^n$ $\|\vec{x}\underset{④.1}{\otimes}\vec{y}\|_F = \|x\|_{\ell_2}\|y\|_{\ell_2}$

☐ $x\otimes y = xy^T$ LHS= $\sqrt{\sum_{ij} x_iy_j^2} = \sqrt{\sum x_i^2 \sum y_j^2}$ = RHS

1c) energy preserve in orthogonal transform 对 A and P为 orthogonal $\|AP\|_F = \|A\|_F$

☐ $\|AP\|_F^2 = tr(APP^TA) = tr(AA^T) = \|A\|_F$

d) AB Frobenius orthogonal iff $tr(AB^T)=0$ 此时 $\|A+B\|_F^2 = \|A\|_F^2+\|B\|_F^2$

☐ $tr((A+B)(A^T+B^T)) = tr(AA^T+AB^T+BA^T+BB^T) \overset{P2.1}{=} tr(AA^T)+\overset{0}{\underline{tr(AB^T)+tr(BA^T)}}+tr(BB^T)$

e) 对 $x\perp y$, ∀a,b 有 $\underset{=0}{xa^T}$ Frobenius orthogonal yb^T

☐ $tr((yb^T)^Txa^T) = tr(by^T\cancel{x}a^T) = 0$

f) $\|A\|_F = \sqrt{\sum_{i=1}^{\min(n,m)}\sigma_i^2(A)}$ ☐ SVD $tr(A^TA) = tr(V\Sigma^T\Sigma V^T) \overset{cyclic}{=} tr(\Sigma^T\Sigma) = \sum\sigma_i^2(A)$

(Ax=b err 在A & b)

应用1 对 perturbed $(A+\overset{matrix}{\boxed{\delta A}})\hat{x} = b + \overset{vec}{\boxed{\delta b}}$, 有 $\frac{\|\hat{x}-x\|}{\|x\|} \le \frac{cond(A)}{(1-cond(A)\frac{\|\delta A\|}{\|A\|})}\left(\frac{\|\delta b\|}{\|b\|}+\frac{\|\delta A\|}{\|A\|}\right)$

δA [illegible] $\hat{x}$为解

☐ 有 $A^{-1}(A+\delta A)\hat{x} = A^{-1}(b+\delta b)$, $\hat{x}+A^{-1}\delta A\hat{x} = x+A^{-1}\delta b$, $\hat{x}-x = A^{-1}\delta b - A^{-1}\delta A(\hat{x}-x) - A^{-1}\delta Ax$

取‖·‖ $\Longrightarrow \|\hat{x}-x\| \overset{def}{\le} \|A^{-1}\|\cdot\|\delta b\| + \|A^{-1}\|\cdot\|\delta A\|\cdot\|\hat{x}-x\| + \|A^{-1}\|\cdot\|\delta A\|\cdot\|x\|$

$\le cond(A)\frac{\|\delta b\|}{\|b\|}\underbrace{\frac{\|b\|}{\|A\|}}_{\le\|x\|} + cond(A)\frac{\|\delta A\|}{\|A\|}\|\hat{x}-x\| + cond(A)\frac{\|\delta A\|}{\|A\|}\cdot\|x\|$

要>0 移项

$(1-condA\frac{\|\delta A\|}{\|A\|})\frac{\|\hat{x}-x\|}{\|x\|} \le cond(A)\left(\frac{\|\delta b\|}{\|b\|}+\frac{\|\delta A\|}{\|A\|}\right)$

应用2. 对 x^* st. $\|Ax^*\|_2 \le \|Av\|_2$, ∀v, (x^*, v unit) 对 $\Delta = \inf_{A+\delta A\ singular}\|\delta A\|$, 有 $\Delta = \frac{1}{\|A^{-1}\|}$

☐ 1° inf 可取到: ∃δA 满足 $\|\delta A\| = \frac{1}{\|A^{-1}\|}$ .st. A+δA singular

P51.下有 $V_S = V\oplus V^{\perp}$, 令 $z = \alpha x^* + w$, 其中 $x^* \in V$, $w\in V^{\perp}$, ie. $x^*\perp w$

Consider $\delta A = -yx^{*T}$, where $y = Ax^*$ $\|\delta A\| \overset{def\ of\ norm}{=} \max_{\|z\|=1}\|yx^{*T}z\| = \max_{\|\alpha x^*+w\|=1}\|y\underbrace{x^{*T}\alpha x^*}_{=1\ 因\ unit} + y\underbrace{x^{*T}w}_{=0}\|$

w不影响‖y‖ 因为只有 αx^* 的长度; ∴ w=0

$= \max_{\|\alpha x^*+w\|=1}\|\alpha y\| = \|y\|$ 而‖x*‖=1, ∴ α=1 , 而 $\|A^{-1}\| = \max_{y\ne0}\frac{\|A^{-1}y\|}{\|y\|} = \max_{x\ne0}\frac{\|A^{-1}Ax\|}{\|Ax\|} = \max_{x\ne0}\frac{\|x\|}{\|Ax\|} \overset{分子为1}{=} \max_{\|x\|=1}\frac{1}{\|Ax\|} \overset{即求\|Ax\|最小}{=} \frac{1}{\|Ax^*\|} = \frac{1}{\|y\|}$

$\therefore = \frac{1}{\|A^{-1}\|}$

且, A+δA singular: 对 x^*, $Ax^*+\delta Ax^* = y - yx^{*T}x^* = 0$, ∴ nullity≠0,

2° $\frac{1}{\|A^{-1}\|}$ smallest, 即 $\frac{1}{\|A^{-1}\|} \le ∀\|\delta A\|$ consider $\underline{z}$ st. $(A+\delta A)z = 0$ (以z因 singular), $Az = -\delta Az$, $z = -A^{-1}\delta A\cdot z$ $\|z\| \le \|A^{-1}\|\cdot\|\delta A\|\cdot\|z\|$

$\frac{1}{\|A^{-1}\|} \le \|\delta A\|$

norm 4

待证明 $\|A\| = ?$ i. $\forall \|x\|=1, \|Ax\| \le ?$
ii. constant $\|\hat{x}\|=1$ st. $\|A\hat{x}\| = ?$ (obtainable)

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max Σ row

例1. $\|\vec{x}\|_\infty = \max_i |x_i|$ induced matrix norm 为 $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$

☐ $P_{49.4}$ 取 $\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty$，令 $\|Ax\|_\infty = \max_i \underbrace{\left|\sum_k a_{ik} x_k\right|}_{\le \sum |a_{ik}||x_k|} \le \max_i \sum_k |a_{ik}|$. 找 $\|x\|_\infty = 1$ 满足等号即可

令有 $\sum_{k=1} |a_{jk}| = \max_i \sum_k |a_{ik}|$ 令 $x_k = \begin{cases} \frac{|a_{jk}|}{a_{jk}} & a_{jk} \neq 0 \\ 1 \end{cases}$ 因最大1, $\|x\|_\infty = 1$. 且对 $i \neq j$ $|\sum_k a_{ik} x_k| \le \sum_k |a_{ik}| \le \max_i \sum_k |a_{ik}| = \sum_k |a_{jk}|$ 等号取到

对 $|\sum a_{jk} x_k| = \sum_k |a_{jk}|$

例2 $\|\vec{x}\|_{\ell_1} = \sum |x_i|$ induced matrix norm $\|A\|_1 = \max_k \sum_{i=1} |a_{ik}|$ max Σ col

☐ 同例1. $\|A\| = \sup_{\|x\| \le 1} \|Ax\| = \sup \|\sum_k a_k x_k\| \le \sup \sum_k |x_k| \|a_k\| \le (\max_k \|a_k\|) \cdot \sup \sum_k |x_k| \le \max_k \sum_i |a_{ik}|$ 取 $x_k = \begin{cases} 1 & k=j \\ 0 \end{cases}$，设有 $\sum_i |a_{ij}| = \max_k \sum_i |a_{ik}|$

因, 5 , n△不等式 提出 $\|a_k\|$ 最大 $\|x\|_{\ell_1} = 1$

例3. 对 $\|A\| = 5$, $\|A^{-1}\| = 2$ 若 $Ax = b$ 求 计算到 $\hat{x}$, 希望 $\|A\hat{x} - b\| = 10^{-6}$, $\|\hat{x} - x\|$ 应如何?

☐ $10^{-6} = \|A\hat{x} - Ax\| \le \|A\| \cdot \|\hat{x} - x\|$, $\therefore \|\hat{x} - x\| \ge \frac{10^{-6}}{5}$; 而 $\|\hat{x} - x\| = \|A^{-1}(A\hat{x} - Ax)\| \le \|A^{-1}\| \|A\hat{x} - b\| = 2 \times 10^{-6}$

bound $\frac{10^{-6}}{5} \le \|\hat{x} - x\| \le 2 \times 10^{-6}$

例4 norm equiv 性 T(i) , 推导之
i) $\|x\|_\infty \le \|x\|_2 \le \sqrt{n} \|x\|_\infty$
(ii) $\|x\|_\infty \le \|x\|_1 \le n \|x\|_\infty$

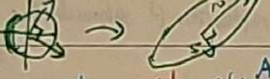
☐ i) $\max |x_i| = \sqrt{(\max |x_i|)^2} \le \underbrace{\sqrt{|x_1|^2 + \cdots + |x_n|^2}}_{\|x\|_2} \le \sqrt{n \times (\max |x_i|)^2}$

ii) $\max |x_i| \le \underbrace{|x_1| + \cdots + |x_n|}_{\|x\|_1} \le n \cdot \max |x_i|$

例4 用 induced norm/operator norm def 估计 $\|A\|_{\ell_2}$
i) $\begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$
ii) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ 因 pd 又 (Strang P_k)
iii) $\begin{pmatrix} 2\cos\theta & -2\sin\theta \\ 2\sin\theta & 2\cos\theta \end{pmatrix}$

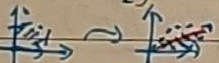
$\boxed{\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}}$

☐ i) eval 5 $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$
10 $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$



最大 stretch 是 10. $\therefore \|A\|_2 = 10$

ii)



考虑对角线, $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{A} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\therefore$ 至少 $\|A\|_2 \ge \frac{\|\begin{pmatrix} 2 \\ 1 \end{pmatrix}\|_2}{\|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\|_2} = \frac{\sqrt{5}}{\sqrt{2}} \approx 1.58$ 而实际为 $1.62 = \sigma_1$

iii) rotating stretch by factor 2 , $\therefore \|A\|_2 = 2$

应用1. 用 $\hat{x}$ approx. $Ax = b$ 有 $\frac{\|x - \hat{x}\|}{\|\hat{x}\|} \le \underbrace{\|A^{-1}\| \cdot \|A\|}_{\text{cond number}} \cdot \frac{\|A - \hat{A}\|}{\|A\|}$, where $\hat{A}$ 为 A 的 perturb

☐ $Ax + (\hat{A} - A)\hat{x} + A(\hat{x} - x) = b$ 有 $\hat{x} - x = -A^{-1}(\hat{A} - A)\hat{x}$. $\|\hat{x} - x\| \overset{\text{minimum property}}{\le} \|A^{-1}\| \|A - \hat{A}\| \|\hat{x}\|$

$\therefore \underset{\text{sol relative error}}{\text{LHS}} \le$ cond. number $\cdot \underset{\text{data relative error}}{\frac{\|A - \hat{A}\|}{\|A\|}}$

cond(A) 过大 有 ~~[illegible]~~ instable

Note $\|A^{-1}\| \cdot \|A\| \ge \|A^{-1} A\| = 1$, 且 $\forall M$, $\exists A$ st. cond(A) $\ge M$

☐ consider $\varepsilon < 1/M$ 对 $A = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$ 有 $\|A\|_{\ell_1} = 1$ st. cond(A) $= 1/\varepsilon > M$

$A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\varepsilon} \end{pmatrix}$ $\|A^{-1}\|_{\ell_1} = 1/\varepsilon$

例5 若 $\|x\|_V = \|\overset{\text{可逆}}{V} x\|$, where $\|\cdot\|$ any norm 易证 $\|\cdot\|_V$ 亦为 norm. 证 induced $\|A\|_V = \|VAV^{-1}\|$

☐ 1° WTS $\forall \|x\|_V = \|Vx\| = 1$ 有 $\|Ax\|_V = \|VAV^{-1}\|$. 但由 $\|VAV^{-1}\|$ def. $\forall \|z\| = 1$ 有 $\|VAV^{-1}z\| \le \|VAV^{-1}\|$ 因此选 $\overset{\|z\|=1}{z = Vx}$ 有 $\|VAV^{-1}Vx\| = \|VAx\| = \|VAV^{-1}\|$

2° 由 $\|VAV^{-1}\|$ def 有 $\|VAV^{-1}z^*\| = \|VAV^{-1}\|$, 令 $x^* = V^{-1}z^*$, 有 $\|x^*\|_V = \|z^*\| = 1$ 且 $\|Ax^*\| = \|VAV^{-1}z^*\| = \|VAV^{-1}\|$

证1. cond(A) $= \max_{\|x\| = \|y\|} \frac{\|Ax\|}{\|Ay\|}$ ☐ 对 $\|A\| \cdot \|A^{-1}\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \cdot \max_{z \neq 0} \frac{\|A^{-1}z\|}{\|z\|} = \max_{\|x\|=1} \|Ax\| \cdot \max_{\|y\|=1} \frac{\|y\|}{\|Ay\|} = \max_{\|x\|=\|y\|} \frac{\|Ax\|}{\|Ay\|}$

P46.1 def, sup 可取到 (P49.4.NOTE)
取 $x \leftarrow \frac{x}{\|x\|}$ $x \neq 0$, 此时 有 $\|x\| = 1$ 即 x y 同 norm
又令 $y = Az$, 因 A 可逆, nullity $= 0$, $y \neq 0$, 用 $y \leftarrow \frac{y}{\|y\|}$

2. ~~对~~ perturbed $\tilde{x}$ ($A\tilde{x} = \tilde{b}$) 有 $\frac{\|b - \tilde{b}\|}{\|b\|} \frac{1}{\text{cond}(A)} \le \frac{\|x - \tilde{x}\|}{\|x\|} \le \text{cond}(A) \frac{\|b - \tilde{b}\|}{\|b\|}$ svd 非常好 见后中

☐ 左 $\frac{\|A\| \cdot \|x - \tilde{x}\|}{\|A\| \cdot \|x\|} \ge \frac{\|Ax - A\tilde{x}\|}{\|A\| \cdot \|A^{-1}b\|} \ge \frac{\|b - \tilde{b}\|}{\|A\| \cdot \|A^{-1}\| \cdot \|b\|}$ 右 $\frac{\|A\| \cdot \|x - \tilde{x}\|}{\|A\| \cdot \|x\|} \le \frac{\|A\| \cdot \|A^{-1}\| \|b - \tilde{b}\|}{\|Ax\|}$

3. cond(A) (in 2-norm) $= \begin{cases} \frac{\lambda_{max}}{\lambda_{min}}, & A \text{ pd.} \\ \frac{\sigma_1}{\sigma_n}, & A \text{ 若方阵} \\ 1, & A \text{ unitary} \end{cases}$ ☐ Psvd证5

$A^T A$ pd, eval $\sigma_1^2 \ge \cdots \ge \sigma_n^2 > 0$ $\therefore \|A\|_2 = \sigma_1$
$(A^T A)^{-1}$ 即 $1/\sigma_n^2 \ge \cdots \ge 1/\sigma_1^2 > 0$ $\|A^{-1}\|_2 = 1/\sigma_n$

Psvd T_5 & P_{vii}

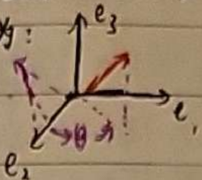
Rotation

Date.　　/

回例2

1. 一般只考虑逆时针，如2D $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

构建3D rotate：WLOG 令逆时针为：



rotate around e_1 axis $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

e_2: $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

e_3 $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. P 对 rotation matrix R

1) (preserve distance) $||x-y|| = ||Rx-Ry||$

□ 由 4) & Pss. P_1 ii)

2) (preserve angle) $\cos(x,y) = \cos(Rx,Ry)$

□ 易证 $= \frac{x^Ty}{||x||\cdot||y||}$ 而 $\frac{x^TR^TRy}{||Rx||\cdot||Ry||} = \frac{x^Ty}{||x||\cdot||y||}$

3) 非 commute ∴ rotate order matter，但 2D Rotation commutative

4) rotation 为 orthogonal

□ 如 2D $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $Q^TQ = I$

5) $|\det R| = 1$ □ 由 4) & Pss. P_1 　det 为 linear T 后的 area 之比原 area 　回. T. rotate 不 scale area

3. Householder Reflection

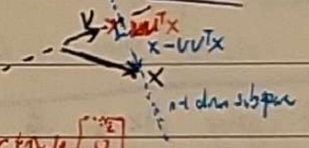
Gram 非 num stable (cond(A) large 时 至少 1个 col $a_j \overset{\text{almost}}{\in} \langle a_{1..j-1}\rangle$，∴ Gram 的 $a_j - \sum_i \frac{\langle u_i, a_j\rangle}{\langle u_i, u_i\rangle} u_i$ 会致 close num (numeric P11.3) ∃ cancel error　(P50)

∴ 提出 stable alg：reflect a_j s.t. $a_j \in \langle u_{1..j-1}\rangle^{\perp}$

1° def $P_n = I - 2uu^T$ where $||u||_2 = 1$，称 reflection P 为 householder Transform

P_2 i) P orthogonal $P^T = I - 2vv^T = P$；$P^TP = P^2 = I - 4vv^T + 4\underbrace{(vv^T)(vv^T)}_{4v(v^Tv)v^T,\ \text{norm}=1} = I$

ii) P reflect x to n-1 dim subspace which ⊥ v



□ 因 $v(v^Tx) = \text{proj}_v x$ 而 $Px = x - vv^Tx$ 见图

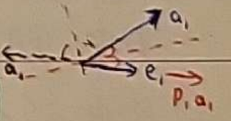
2° (QR decomp) 想找 reflection $\{P_i\}$ s.t. $P_n \cdots P_1 A = R$，选择 $Q = P_1^T \cdots P_n^T$，每个 P_i 只造 R 的 1 col　upper triangle $\begin{bmatrix} \ddots & \\ 0 & \end{bmatrix}$

即 $P_1A = \begin{pmatrix} r_{11} & \vdots & \vdots \\ 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix}$，$P_2P_1A = \begin{pmatrix} r_{11} & r_{12} & \vdots \\ 0 & r_{22} & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix}$ $P_3P_2P_1A = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \vdots \\ \vdots & r_{22} & r_{23} & \vdots \\ & \vdots & r_{33} & \vdots \end{pmatrix}$　Pss. P_2 (iii)

2种 reflect a_1 ① ② 取 r_{11}

1° P_1: $P_1\vec{a}_1 = r_{11}\vec{e}_1 = a_1 - 2v_1v_1^Ta_1$，$v_1 = \frac{a_1 - r_{11}e_1}{2v_1^Ta_1}$，而 $||P_1a_1||_2 = ||a_1||_2 = ||r_{11}e_1|| = |r_{11}|$

depend on 1st ele of a_1　　近似 因 $||v_1||_2 = 1$



∴ pick $r_{11} = -\text{sign}((\vec{a}_1)_1)\cdot||a_1||_2$，$v_1 \simeq \frac{a_1 - r_{11}e_1}{||a_1 - r_{11}e_1||_2}$　denote $P_1A = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ 0 & & & & \\ \vdots & a_2 & & A_2 & \end{pmatrix}$

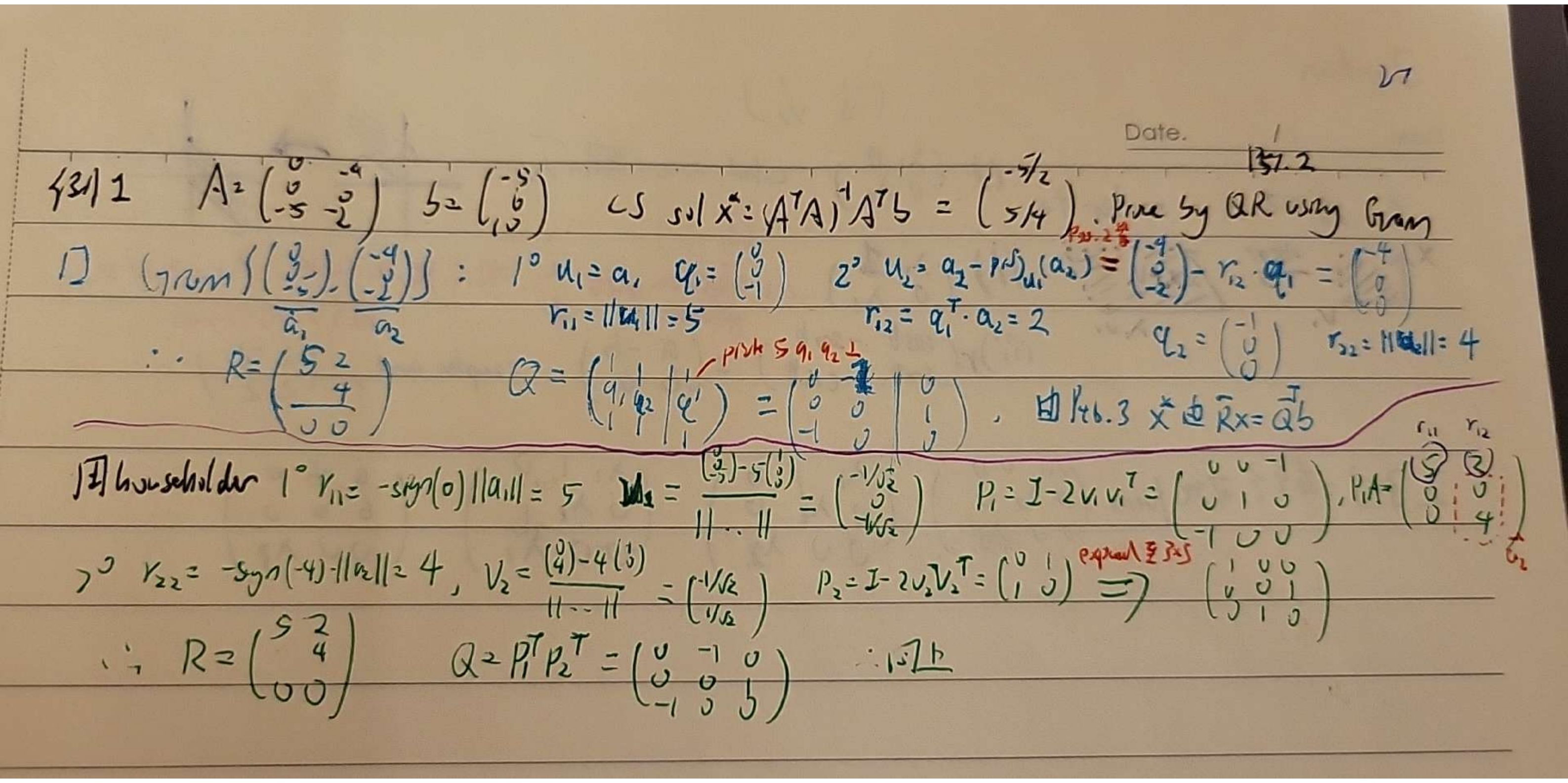
2° P_2: 因 $\begin{cases} P_2(P_1A)_{x1} = \begin{pmatrix} r_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (I - 2v_2v_2^T)\cdot \begin{pmatrix} r_{11} \\ \vdots \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ * \\ \vdots \\ * \end{pmatrix} \\ P_2(P_1A)_{x2} = \begin{pmatrix} r_{12} \\ r_{22} \\ 0 \\ \vdots \end{pmatrix} \quad \begin{pmatrix} r_{12} \\ a_2 \end{pmatrix} \Rightarrow 令 I - 2v_2v_2^T = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & sth & \\ 0 & & & \end{pmatrix}，而之 \Rightarrow \begin{pmatrix} r_{12} \\ sth\cdot\vec{a}_2 \end{pmatrix} \quad \begin{pmatrix} r_{22} \\ 0 \\ \vdots \\ 0 \end{pmatrix} 类似 1° \end{cases}$

~~是否 $P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & sth & \\ 0 & & \end{pmatrix}$~~

for sth : → ∴ $r_{22} = -\text{sign}((\vec{a}_2)_1)\cdot||a_2||_2$，$v_2 \simeq \frac{a_2 - r_{22}e_1}{||a_2 - r_{22}e_1||_2}$　denote $P_2P_1A = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \cdots \\ 0 & r_{22} & r_{23} & \cdots \\ \vdots & \vdots & a_3 & A_3 \end{pmatrix}$

3° 同理 $r_{33} = -\text{sign}((\vec{a}_3)_1)\cdot||a_3||_2$，$v_3 \simeq \frac{a_3 - r_{33}e_1}{||a_3 - r_{33}e_1||_2}$

…… till n°



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157.2

例1 $A=\begin{pmatrix}0 & -4\\ 0 & 0\\ -5 & -2\end{pmatrix}$ $b=\begin{pmatrix}-5\\ 6\\ 10\end{pmatrix}$ LS sol $x^*=(A^TA)^{-1}A^Tb=\begin{pmatrix}-5/2\\ 5/4\end{pmatrix}$. Prove by QR using Gram

P33.2有

[1] Gram $\{\underbrace{\begin{pmatrix}0\\ 0\\ -5\end{pmatrix}}_{a_1},\underbrace{\begin{pmatrix}-4\\ 0\\ -2\end{pmatrix}}_{a_2}\}$: 1° $u_1=a_1$ $q_1=\begin{pmatrix}0\\ 0\\ -1\end{pmatrix}$ 2° $u_2=a_2-\text{proj}_{u_1}(a_2)=\begin{pmatrix}-4\\ 0\\ -2\end{pmatrix}-r_{12}\cdot q_1=\begin{pmatrix}-4\\ 0\\ 0\end{pmatrix}$

$r_{11}=\|u_1\|=5$ $r_{12}=q_1^T\cdot a_2=2$

$q_2=\begin{pmatrix}-1\\ 0\\ 0\end{pmatrix}$ $r_{22}=\|u_2\|=4$

$\therefore R=\begin{pmatrix}5 & 2\\ & 4\\ \hline 0 & 0\end{pmatrix}$ $Q=\left(\begin{array}{cc|c} q_1 & q_2 & q'\end{array}\right)=\left(\begin{array}{cc|c}0 & -1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\end{array}\right)$, 由P46.3 x^* 由 $\bar{R}x=\bar{Q}^Tb$

pick与 $q_1\ q_2\ \perp$

[2] householder 1° $r_{11}=-\text{sign}(0)\|a_1\|=5$ $v_1=\dfrac{\begin{pmatrix}0\\ 0\\ -5\end{pmatrix}-5\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}}{\|\cdots\|}=\begin{pmatrix}-1/\sqrt{2}\\ 0\\ -1/\sqrt{2}\end{pmatrix}$ $P_1=I-2v_1v_1^T=\begin{pmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix}$, $P_1A=\begin{pmatrix}\overset{r_{11}}{5} & \overset{r_{12}}{2}\\ 0 & 0\\ 0 & 4\end{pmatrix}$ a_2

2° $r_{22}=-\text{sign}(-4)\cdot\|a_2\|=4$, $v_2=\dfrac{\begin{pmatrix}0\\ 4\end{pmatrix}-4\begin{pmatrix}1\\ 0\end{pmatrix}}{\|\cdots\|}=\begin{pmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$ $P_2=I-2v_2v_2^T=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\xRightarrow{\text{expand 至 }3\times3}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$

$\therefore R=\begin{pmatrix}5 & 2\\ & 4\\ 0 & 0\end{pmatrix}$ $Q=P_1^TP_2^T=\begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\end{pmatrix}$ ∴同上

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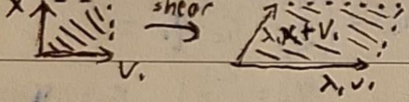
$\begin{pmatrix} -1 & 0 \\ -1 & 1/2 \end{pmatrix}$

1. 每 2×2 A similiar to i) $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ L.Ind. evec 图. T_{2a}



$\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}$

ii) $\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$ λ_1 algebraic multi 2, geometric 1 找个 ~~x~~, v_1 L.Ind, $Ax = \lambda_1 x + v_1$

∴用$\{v_1, x\}$作basis

iii) $r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ complex evel, $\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$

2. 3×3 有4个可能form $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$ $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$ $\begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

例1. $\begin{cases} x_1+x_2+2x_3+x_4=1 \\ x_1-2x_2 \qquad -x_4=-2 \end{cases}$ 用右.2、

因 $rk\begin{bmatrix}1&1&2&3\\1&-2&0&-1\end{bmatrix}=rk\begin{bmatrix}1&1&2&3&1\\1&-2&0&-1&-2\end{bmatrix}$，∴ ∃ sol. assign x_3, x_4 random，有 $\begin{bmatrix}1&1\\1&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1-2x_3-x_4\\-2+x_4\end{bmatrix}$

thus $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ 可为 $\begin{bmatrix}1&1\\1&-2\end{bmatrix}^{-1}\begin{bmatrix}1-2x_3-x_4\\-2+x_4\end{bmatrix}=\begin{bmatrix}-\frac{1}{3}x_3-x_4/3\\1-\frac{2}{3}x_3-\frac{2}{3}x_4\end{bmatrix}$ general sol $\begin{bmatrix}x_1\\ \vdots\\x_4\end{bmatrix}=\begin{bmatrix}\vdots\\x_3\\x_4\end{bmatrix}$

应用 (minus 1 trick) 解 $\overset{Ax=0}{eq}$ 时化为RREF，并在missing pivot处加 $(0\cdots-1\cdots0)$，sol为右-1的col

如 回.Ex2 $A=\begin{pmatrix}1&0&9&-8\\0&1&-4&5\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$ 解为 $a\begin{pmatrix}9\\-4\\-1\\0\end{pmatrix}+b\begin{pmatrix}-8\\5\\0\\-1\end{pmatrix}$

1. Gaussian Elimination 不stable.

eg. $\left(\begin{array}{cc|c}\varepsilon&1&1\\1&1&2\end{array}\right)\Rightarrow\begin{cases}x=\frac{1}{\varepsilon}(1-y)\approx1 \quad \text{cancel error}\\ y=\frac{2\varepsilon-1}{\varepsilon-1}\approx1\end{cases}$ 问题在于 ε pivot 与其余entry相比太小

∴ sol为不用此pivot：⇒ $\left(\begin{array}{cc|c}1&1&2\\ \varepsilon&1&1\end{array}\right)\Rightarrow\begin{cases}x=2-y\approx1\\ y=\frac{1-2\varepsilon}{\varepsilon-1}\approx1\end{cases}$

∴ 据此为启发，impl Gaussian Elimination w/ pivoting，这样大多数情况 stable

例2 partial pivot
$\left(\begin{array}{cccc|c}6&-2&2&4&16\\12&-8&6&10&26\\3&-13&9&3&-19\\-6&4&1&-18&34\end{array}\right)\Rightarrow\left(\begin{array}{cccc|c}12&-8&6&10&16\\6&-2&2&4&26\\3&\vdots&\vdots&&\vdots\\-6&\vdots&\vdots&&\vdots\end{array}\right)\sim\left(\begin{array}{cccc|c}12&-8&6&10&26\\0&2&-1&-1&3\\0&-11&&&\vdots\\0&0&4&&\vdots\end{array}\right)\Rightarrow\begin{pmatrix}\vdots&-8&\\ &-11&\\ &2&\cdots\\ \vdots&0&\end{pmatrix}$

max abs = row2, swap　　max abs = row3, swap

$\sim\begin{pmatrix}\vdots&-8&6&\vdots\\ &-11&15/2&\\ &0&\cancel{4/11}&\\ &0&4&\vdots\end{pmatrix}\Rightarrow\cdots\Rightarrow\left(\begin{array}{cccc|c}12&-8&6&10&26\\0&-11&15/2&1/2&25/2\\0&0&4&-13&-21\\0&0&0&3/11&3/11\end{array}\right)$ sol 同方程系数消元

swap row3&4

Note 此处 partial pivot swap along process，若预先知道哪几row swap，先swap再消元 sol 同边消元边swap，即先permute

2. 对 $Ax=b$，$A_{mn}=\begin{pmatrix}\bar A_{nn}\\0\end{pmatrix}$ $(m\geq n)$，$b=\begin{pmatrix}c_n\\d\end{pmatrix}$ 的 least square sol 同 $\bar Ax=c$，因：　若为 $\begin{pmatrix}c_n\\d_{m-n}\end{pmatrix}$

P: orthogonal. 作法: change standard basis → col(A) — orthonormal basis (用Gram) P55 T6

若设A非此form，可change of basis $\underline{P}$ 令 $PA=\begin{pmatrix}\bar A\\0\end{pmatrix}$，有 $\|b-Ax\|_2=\|P(b-Ax)\|_2=\|Pb-\begin{bmatrix}\bar Ax\\0\end{bmatrix}\|_2$
$=\|\begin{pmatrix}c-\bar Ax\\d\end{pmatrix}\|_2=\|c-\bar Ax\|_2+\|d\|_2$，min when $\bar Ax=c$

eg. $A=\begin{pmatrix}1&0&1\\2&3&5\\5&3&-2\\3&5&4\\-1&6&3\end{pmatrix}$ $b=\begin{pmatrix}4\\-2\\5\\-2\\1\end{pmatrix}$ 故 $R(A)=\left\langle\begin{pmatrix}1\\2\\5\\3\\-1\end{pmatrix}\begin{pmatrix}0\\3\\3\\5\\6\end{pmatrix}\begin{pmatrix}1\\5\\-2\\4\\3\end{pmatrix}\right\rangle\perp N(A^T)=\left\langle\begin{pmatrix}23\\-19\\6\\15\\0\end{pmatrix}\begin{pmatrix}84\\-27\\-3\\0\\15\end{pmatrix}\right\rangle$

$\xRightarrow{\text{Gram}}$ q_1 q_2 q_3　　$\xRightarrow{\text{Gram}}$ p_4 p_5

P51.3 $\mathbb{R}^5=R(A)\oplus N(A^T)$, basis $\{p_{1\ldots5}\}$

$\underset{e\to p}{P}=\underset{p\to e}{P^T}=\begin{bmatrix}-p_1-\\ \vdots\\ -p_5-\end{bmatrix}$　P orthogonal

$PA=\begin{pmatrix}6.3246&4.7&1.5\\ \vdots&7.51&5.2\\ \vdots&&4.2\\0&0&0\\0&0&0\end{pmatrix}$ $\tilde A$

有 $Pb=\begin{pmatrix}2.84\\-1.13\\-3.92\\2.06\\4.58\end{pmatrix}$ c

NOTE 实不能记算 $N(A^T)$，只用 $P=\begin{pmatrix}-q_1-\\ \vdots\\ -q_3-\\ *\\ *\end{pmatrix}=\begin{bmatrix}\tilde P\\ sth\end{bmatrix}$

而 $PA=\begin{bmatrix}\tilde PA\\0\end{bmatrix}$，∴ 实为解 $\tilde PAx=\tilde Pb$

有 $\tilde A$ 上△ $\tilde PA=\tilde A=\begin{pmatrix}-p_1-\\ \vdots\\ -p_n-\end{pmatrix}\begin{pmatrix}a_1&\cdots&a_n\end{pmatrix}=\begin{pmatrix}p_1^Ta_1&\cdots&p_1^Ta_n\\ \vdots&\ddots&\vdots\\ p_n^Ta_1&\cdots&p_n^Ta_n\end{pmatrix}$ 附P为 basis for $N(A^T)$，∴ $A^Tq_4=0$，$0^T=q_4^TA$

可见实为QR decomp：P57.2 $PA=R=\begin{bmatrix}\bar R=\bar A\\0\end{bmatrix}$，$A=P^{-1}R=P^TR=QR$　=0 by Gram process

3. (LS w/QR) 由 Q^T 不变 L_2 norm P55 P11

$\|Ax-b\|=\|QRx-b\|=\|Rx-Q^Tb\|=\|\begin{bmatrix}\bar R\\0\end{bmatrix}x-\begin{bmatrix}\bar Q^T\\sth\end{bmatrix}b\|=\|\begin{bmatrix}\bar Rx-\bar Q^Tb\\sth\end{bmatrix}\|=\|\bar Rx-\bar Q^Tb\|+\|sth\|$ 不可变 不依赖x

min 当 $\bar Rx=\bar Q^Tb$ 取到

Sys of Equations 1

Date. /

1. 对 $A_{m\times n}$，$\boxed{1}.\underset{J}{A}x=b$ ∃ sol $\Leftrightarrow$ $rkA = rk[A\mid b]$

□ $\Rightarrow$ b = L. comb of A's col $=\sum_{i=1}^{n} x_i a_i$, thus $b\in\langle a_1\cdots a_n\rangle$, $\underline{rkA=\dim\langle a_1\cdots a_n\rangle}$ $\overset{\boxed{23}.J_{1b}}{}$ $= \underline{\dim\langle a_1\cdots a_n, b\rangle}$ $\overset{P_2.P_{V)}}{}$ $= rk[A\mid b]$

$\Leftarrow$ 令 $rk=r$ 则 A有 r L.ind col $a_1\cdots a_r$，$[A\mid b]$ 也有同样 r 个，∴ b 可由 $a_1\cdots a_n$ L.comb $\boxed{21}J_{1b}$

∴ $b=\sum_{i=1}^{n} x_i a_i$，weight x_i 即 $\vec{x}$ sol

NOTE, x unique $\Longleftrightarrow$ $rkA = rk[A\mid b] = n$ 可逆

□ $\Rightarrow$ 令 $rk<n$，则有 $y\neq 0$ st $\underset{\sim\sim\sim}{\overset{\text{col L.dep}}{Ay=0}}$，若 x 为 sol，$x+y$ 也 sol，矛盾

$\Leftarrow$ $\underline{\exists \text{ col}}^{1.}$，若非 unique，$Ax=Ay=b$，$A\overset{\neq 0}{(x-y)}=b$ 可 A's col L.ind 矛盾 OR，$\underline{A\text{可逆}}$ 即 $\boxed{1}.4$ $\boxed{2}.1$ $\therefore x=A^{-1}b$

2. $A_{m\times n}$，$n\geq m$，$rkA=m$． $Ax=b$ 的 sol 为：assign $n-m$ 个 random，剩余 m 个 以此 solve

□ 令 $\overset{m\text{个col}}{\widetilde{a_1\cdots a_m}}$ L.ind $Ax=b \rightarrow x_1a_1+\cdots+x_ma_m = b - x_{m+1}a_{m+1}-\cdots-x_na_n$，assign $\begin{cases} x_{m+1}\leftarrow \varphi_{m+1}\\ \vdots \\ x_n\leftarrow\varphi_n\end{cases}$，$\varphi_i$ random

∴ $\overset{B,\text{可逆}}{[a_1\cdots a_m]}\begin{bmatrix}x_1\\ \vdots\\ x_m\end{bmatrix} = \cdots$，$\begin{bmatrix}x_1\\ \vdots\\ x_m\end{bmatrix} = B^{-1}[b-\varphi_{m+1}a_{m+1}-\cdots-\varphi_na_n]$

代数复习 P116 例3 / LA.PS2.2.2°

3. (least square) $\min_x f(x) = \|b-Ax\|_2^2$ 当 A_{mn} 有

- $m=n$ 且 A invertible. unique sol $x^*=A^{-1}b$，$f(x^*)=0$

- $m>n$，$f(x^*)>0$，有 normal eq $A^TA x^* = A^Tb$ 见 Puta P11.3 note 当 $m=n$ x^* 也 成立 $=A^{-1}b$

  $\boxed{1}$. 证1 $rkA^TA = rkA = n$ $\quad$ ~~inv~~ matrix P123. P86. 例3 $=A^+$

  - A^TA 可逆：$x^* = (A^TA)^{-1}A^Tb = \underline{A^+}b$

  - 否：x^* inf many sol，但有同一 $f(x^*)$

□ 除了 Puta P11.3 取 $\nabla=0$；WTS x^* min $f(x)$ $\Longleftrightarrow$ $A^TAx^*=A^Tb$ by $\underset{\sim\sim\sim}{\text{orthogonal decomposition}}$ P51. J_{2d})

$\Rightarrow$ $b = \overset{\in\text{Col}A}{\underline{c}} + \overset{\in N(A^T)}{\underline{d}}$ 由 P51. J_{2a}，因 $d\perp \overset{Ax\in\text{Col}A}{Ax-c}$ $\overset{}{\text{orthogonal decomp def}}$，$f(x)=\|c+d-Ax\|_2^2 \overset{P51.J_0}{=} \|c-Ax\|_2^2+\|d\|_2^2$，$x^*$ 对 $Ax=c$ 求到

此时 x^* 有 $A^TAx = A^Tc + \underset{d\in N(A^T),\text{ 即 } A^Td=0}{0} = A^T(c+d) = A^Tb$ $\qquad$ $\Leftarrow$ $A^TAx^* = A^T(c+d) = A^Tc$

∴ $A^T(Ax^*-c)=0$，∴ 有 $\underset{\in R(A)}{\widetilde{Ax^*-c}} \in N(A^T)$ $\overset{\sim}{\in K(A)}$ 因 $\mathbb{R}^m = R(A)\oplus N(A^T)$ 唯一 重叠 为 $Ax^*-c=0$

NOTE $\overset{m>n}{}$ 求 pseudo-inverse $\overset{(\text{当A full rk})}{}$：令 $B=A^TA$，$y=A^Tb$，即 求解 $Bx=y$，where B $\underline{pd}$ (Optim. P6. P26) $N(A)=\{0\}$

1° cholesky $\overset{\text{Optim. P7, 1}}{\text{decomp}}$ $B=GG^*$ $\quad$ 2° solve $Gz=y$ by forward sub $\quad$ 3° solve $G^Tx=z$ by backward sub

定理1. (solve normal equation is numerically instable.)

对 nonsquare $A_{mn}(m>n)$ def$^{\text{n}}$ $\underset{\sim\sim}{\overset{\text{对 square 见 P123 定理1}}{\text{cond}_2(A)}} = \|A\|_2\,\|A^+\|_2$ $\qquad$ rel. error bdd by cond(A)

证 $\quad$ 1° $\text{cond}_2(A) = \frac{\sigma_1}{\sigma_n}$ $\quad$ 2° 对 least square：perturb $\tilde{b}$ 的 sol $\tilde{x}$ $\overset{=A^+\tilde{b}}{}$ 有 $\frac{\|x-\tilde{x}\|_2}{\|x\|_2} \leq \text{cond}_2(A)\frac{\|b-\tilde{b}\|_2}{\|c\|_2}$

UV orthogonal，证 P55，J_6

□ 1° SVD $A = U\underset{\sim\sim\text{记为}\begin{bmatrix}S\\0\end{bmatrix}}{\begin{pmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & 0 & \end{pmatrix}}V^T$，∴ $\underline{\|A\|_2 = \|\Sigma\|_2 = \sigma_1}$，而 $A^+ = (V\Sigma^TU^TU\Sigma V^T)^{-1}V\Sigma^TU^T$

$(S^{-1})^{T}[S\ 0] = S^{-1}\cdot S^{-1}[S\ 0]$

$=(V[S\ 0]\begin{bmatrix}S\\0\end{bmatrix}V^T)^{-1}V[S\ 0]U^T = V^{T^{-1}}S^{2^{-1}}V^{-1}V[S\ 0]U^T \overset{(3.12)}{=} VS^{2^{-1}}[S\ 0]U^T = V[S^{-1}\ 0]U^T$，∴ $\|A^+\|_2 = \|[S^{-1}\ 0]\| = \frac{1}{\sigma_n}$

2° 因 $\|x-\tilde{x}\|_2 \leq \|A^+\|\cdot\|b-\tilde{b}\|$ 且 对 $b=c+d$ $Ax=c$，∴ $\|c\|\leq\|A\|\cdot\|x\|$

elementary

(Gaussian Eliminate n×n matrix
$O(n^3)$ 每对 1st row $\boxed{n} \rightarrow \boxed{\begin{smallmatrix}1\\0\\ \vdots\\0\end{smallmatrix}} n-1$
scaling, adjust 都为 operation,
1st row 对各行都有 scaling 再对下方的 n-1 row
adjust ∴ n(n-1) $\sum_{k=1}^{n} k(k+1) = \frac{n^3-n}{3}$
∴ 对 k=1…n, operation 是 k

1. elementary row operation

1) row scaling: 某一row 乘非0 scalar α. $A_{i,:} \rightarrow \alpha A_{i,:}$

2) row exchange: 互换二行 $A_{i,:} \leftrightarrow A_{j,:}$

3) row adjustment: 一row 乘以倍数加于另一row $A_{j,:} \rightarrow A_{j,:} + \alpha A_{i,:}$, $i \neq j$

2. matrix R 为 in row echelon form iff

i) 只有 0 的 row 在 R 底部, ii) 非0 row, first nonzero entry 为 1. (leading entry) iii) ∀2个连续 row, 下方的 leading entry 在右

row echelon form 为 reduced row echelon form iff ∀col 有 leading entry 的其他 position 0

例见 四.1.

3. $A, B \in M_{m,n}(F)$, A 与 row equivalent 即 A 可以 elementary row operation 成 B.

易证 $A \sim B$ 为 equivalent relation

且, T_1: $A \sim B$, 则 $B = \underline{E}A$ (elementary matrix); 因 row operation 同 column operation, $B = AE'$ ($A \sim B$ by col)

□ row operation: 1) $E = \begin{bmatrix}1&&\\&\alpha&\\&&1\end{bmatrix}$ ← i row 2) $E = \begin{bmatrix}1&&&\\&&1&\\&1&&\\&&&1\end{bmatrix}$ 即 $E_{i,:} = e_j^T$, $E_{j,:} = e_i^T$

这样 $(EA)_{i,\alpha} = A_{j,\alpha}$
$(EA)_{j,\alpha} = A_{i,\alpha}$

3) $E = \begin{bmatrix}1&&&\\&\ddots&&\\&\alpha&1&\\&&&1\end{bmatrix}$ (i, j) 这样 $(EA)_{j,\alpha} = \alpha A_{i,\alpha} + A_{j,\alpha}$ 同 3

由 $B_{i,:} = (EA)_{i,:} = E_{i,:}A$ 看出, 且 E 由 A→B 的 op 施加于 I_n 上得到)

对 col operation 1) $E = \begin{bmatrix}1&&\\&\alpha&\\&&1\end{bmatrix}$ (i col) 2) $E = \begin{bmatrix}1&&&\\&&1&\\&1&&\\&&&1\end{bmatrix}$ (i col, j col) 3) $E = \begin{bmatrix}1&&&\\&1&\alpha&\\&&1&\\&&&1\end{bmatrix}$ (i, j)

由 $B_{:,i} = AE_{:,i}$ 看出

易知当 A row[col] → B, A^T 对应 col[row] → B^T □ $B = EA$ [$B = AE$] → $B^T = A^TE^T$ [$B^T = E^TA^T$]

T_2: E (elementary matrix) invertible, 且 inverse 同 type row[col] (相反的 row[col])

□ WLOG, E 由 row op type i 得到, 由 T_1 可视为对 I_n 施加同 op ∴ $I_n \sim E$, ∴ $E \sim I_n$ ∴ 有 E', $I_n = E'E$

思考 prod of elem matrix elem ✗ $\begin{bmatrix}5&\\&1\end{bmatrix}\begin{bmatrix}&1\\1&\end{bmatrix} \rightarrow \begin{bmatrix}&5\\1&\end{bmatrix}$ sum of ‥ elem ✗ $\begin{bmatrix}5&\\&1\end{bmatrix} + \begin{bmatrix}&1\\1&\end{bmatrix} = \begin{bmatrix}5&1\\1&1\end{bmatrix}$

transpose of ‥ elem ✓ type 1, 2 symmetric, type 3 row[col] 为 type 3 col[row] ∴ $A \sim B$ by row[col] ⇒ $A \sim B$ by col[row] ✗ $\begin{bmatrix}1&1\\1&1\end{bmatrix} \sim \begin{bmatrix}5&5\\1&1\end{bmatrix}$ by row, 但不可由col得

$A \sim B$ by row[col] ⇒ $B \sim A$ by row[col] ✓ undo op

T_2 用于求简单的逆矩阵 如 $\begin{bmatrix}&1\\1&\end{bmatrix}^{-1} = \begin{bmatrix}&1\\1&\end{bmatrix}$, $\begin{bmatrix}1&\\&3&\\&&1\end{bmatrix}^{-1} = \begin{bmatrix}1&\\&1/3&\\&&1\end{bmatrix}$, $\begin{bmatrix}1&\\-2&1\end{bmatrix}^{-1} = \begin{bmatrix}1&\\2&1\end{bmatrix}$ (undo 为 2×row)

例 $A = \begin{pmatrix}2&1&1&4\\3&1&2&3\\1&2&1&0\end{pmatrix}$ $B = \begin{pmatrix}2&1&1&4\\-1&-1&0&-5\\1&2&1&0\end{pmatrix}$ $C = \begin{pmatrix}3&5&2&4\\-1&-1&0&-5\\1&2&1&0\end{pmatrix}$ 找到 F st. A = FC

□ A = EB, note 要找 B = ?C 并 st. F = E?, $E = \begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}$, $? = \begin{pmatrix}1&0&-1\\0&1&0\\0&0&1\end{pmatrix}$ ∴ F = E?

Sys Eq^2

Date. /

1. $x_1-2x_2+x_3=0$
$2x_2-8x_3=8$
$-4x_1+5x_2+9x_3=-9$ $\Rightarrow$ coefficient matrix $\begin{bmatrix}1&-2&1\\0&2&-8\\-4&5&9\end{bmatrix}$

$\Updownarrow$ augmented matrix $\left[\begin{array}{ccc|c}1&-2&1&0\\0&2&-8&8\\-4&5&9&-9\end{array}\right]$

$A\cdot X=b$

then, row reduction $\begin{bmatrix}1&-2&1&0\\0&2&-8&8\\-4&5&9&-9\end{bmatrix}\overset{\text{elementary op 回}}{\sim}\begin{bmatrix}1&0&0&29\\0&1&0&16\\0&0&\boxed{1}&3\end{bmatrix}$ echelon 阶梯形 matrix
此为 reduced echelon
每row 1st 1 为 leading entry

例1 $f,g\in\mathbb{R}$ $\begin{cases}x_1+3x_2=f\\cx_1+dx_2=g\end{cases}$ 问为 c d st. consistent (无 ~~any~~ sol 即 inconsistent)

解 $\left[\begin{array}{cc|c}1&3&f\\c&d&g\end{array}\right]\sim\begin{bmatrix}1&3&f\\0&d-3c&g-3cf\end{bmatrix}$ $(d-3c)x_2=\underset{\sim}{g-3cf}\in\mathbb{R}$ ∴应令 $d-3c\neq0$ NOTE: 即 $|A|\neq0$ 时才有唯一 uniq. sol

∴若有 row $[0\cdots0\,|\,b]$ inconsistent, 若 consistent $\begin{cases}\#eq=\#var, & \text{uniq. sol 即 } rkA=n \leftarrow \text{P41.1}\\ \neq, & \infty \text{ sol}\end{cases}$
即 $rk(A|b)=rkA+1$, P47.1.

不同的基代表的东西不一样. $[\vec{i}\ \vec{j}\ \vec{k}]$中的$(3,-1,0)$与$[\vec{u},\vec{v}]$中的$(3,-1)$不一样. 因此坐标中一点应由坐标与基 L.comb.
叫 linear 因固定一个vector不变, 画出的结果为是在一条直线上运动

2. linear combinations. given $\vec{v}_1,\vec{v}_2,\cdots\vec{v}_p$ in $\mathbb{R}^n$, given scalar $c_1,c_2\ldots c_p$,

vector $\vec{y}$ defined by $\vec{y}=c_1\vec{v}_1+c_2\vec{v}_2+\ldots+c_p\vec{v}_p$

$c_1\cdots c_p$: weight, 可写为 $[\vec{v}_1\ \vec{v}_2\cdots\vec{v}_p\ \vec{y}]$

~~Span~~ Span if $\vec{v}_1\cdots\vec{v}_p$ in $\mathbb{R}^n$, set of all linear combinations of $\vec{v}_1\ldots\vec{v}_p$ is denoted by span$\{\vec{v}_1\ldots\vec{v}_p\}=\{c_1\vec{v}_1+\ldots+c_p\vec{v}_p\,|\,\forall c_i\in\mathbb{R}\}$

3. matrix equation $Ax=b$.

$[\vec{a}_1,\vec{a}_2\ldots\vec{a}_n]\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=x_1\vec{a}_1+x_2\vec{a}_2+\ldots+x_n\vec{a}_n$ ($A_{m\times n}$, $X_{n\times1}$
$\vec{a}_n$ 为列向量)

T★ $A\in\mathbb{R}^{m\times n}$, the following statements are equivalent:

a. $\forall b\in\mathbb{R}^m$, $Ax=b$ has a solution (不一定 unique)

b. $\forall b\in\mathbb{R}^m$ is a linear combination of A cols

c. the $\underbrace{\text{columns}}_{\text{vectors}}$ of A span $\mathbb{R}^m$ (当 $n\geq m$时, 见d))

d. $\underline{A}^{\text{非aug }[A|b]}$ has a pivot position in each row

例1. could a set of n vectors in $\mathbb{R}^m$ span all of $\mathbb{R}^m$ when $n<m$?

解 let $A=[v_1\ v_2\ldots v_n]$ theorem d. A cannot ~~has~~ have a pivot position in each row

$\boxed{1}$

Ex $\begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix}$ 1° $\begin{bmatrix} 5\times5 -1 -24 -8 \\ -2\times5 +7 +9 +10 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$

2° $\begin{bmatrix} 5 \\ -2 \end{bmatrix} 5 + \begin{bmatrix} 1 \\ -7 \end{bmatrix}\cdot(-1) + \begin{bmatrix} -8 \\ 3 \end{bmatrix}\cdot 3 + \begin{bmatrix} 4 \\ -5 \end{bmatrix}\cdot(-2) = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$, 即将 $Ax = [\vec{a}_1 \cdots \vec{a}_4]\cdot x = \sum \vec{a}_i\, x_i$

例1) $\{\sin(x), \cos(x), 1\} \subseteq \mathcal{F}(\overset{附2}{\mathbb{R}}, \mathbb{R})$ L.Ind

3D/Wronskian

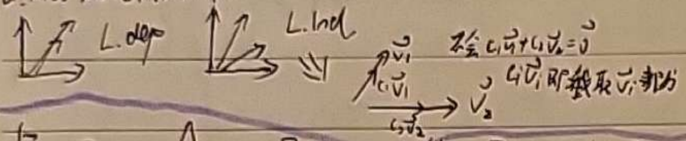
□ $\begin{cases} x=0 & a_2+a_3=0 \\ x=\pi/2 & a_1+a_3=0 \Rightarrow a_1=a_2=a_3=0 \\ x=\pi/4 \end{cases}$

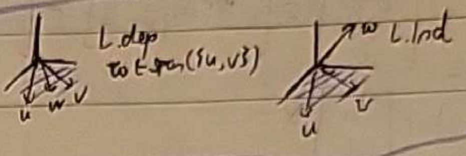
Ex2 describe all sols of $Ax=0$ in parmetric vector form

□ $A = \begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$, $\left[\begin{array}{cccc|c} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 9 & -8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix}$, 非 pivot position $\begin{cases} x_1 = -9x_3 + 8x_4 \\ x_2 = 4x_3 - 5x_4 \\ \to x_3, x_4 \text{ free var} \end{cases}$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3+8x_4 \\ 4x_3-5x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$

L.Ind 的几何解释:





Ex3 A is 3×2 with 2 pivot postions. Do $Ax=0$ has nontrival sol? Do $Ax=b$ has at least one sol for every b?

□ $\left[\begin{array}{cc|c} ▨ & * & 0 \\ 0 & ▨ & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1=0$, $x_2=0$ 1. sol trival

2. no pivot position each row, N.O

Ex4 A 7×5, if columns L.Ind, how many pivot position

□ 5, 这样无 free variable, $\boxed{3}$.F#1

A 5×7, if columns span $\mathbb{R}^5$, how many pivot positions

□ 5, 各行都有一个 pivot position $\boxed{1}$.T

Ex5 let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\vec{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\vec{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, T $\mathbb{R}^2 \to \mathbb{R}^2$ linear, which $\vec{e}_1 \mapsto \vec{y}_1$, $\vec{e}_2 \mapsto \vec{y}_2$. Find images of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (即: 一旦找到 basis 的 map 地址, 可找到 $\forall x \in$ domain 的 map 地址)

□ $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = T(x_1\vec{e}_1 + x_2\vec{e}_2) = x_1\begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 6 \end{bmatrix}$

let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\vec{v}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$, T $\mathbb{R}^2 \to \mathbb{R}^2$ linear, $\vec{x} \mapsto x_1\vec{v}_1 + x_2\vec{v}_2$

find A s.t $T(\vec{x})$ is $A\vec{x}$ for any $\vec{x}$ (即, 找到 $[T]_\beta^\gamma$)

□ $T(\vec{x}) = x\vec{v}_1 + x\vec{v}_2 = [\vec{v}_1\ \vec{v}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}$

见 $\boxed{5}$.1

Ex6 each matrix T is linear. True, $T(\vec{u}+\vec{v}) = \underbrace{A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}}_{\text{matrix 特性}} = T(\vec{u}) + T(\vec{v})$

codomain of T $\vec{x} \mapsto A\vec{x}$ is set of all L.Comb of columns of A False

应为 range. codomain 为 $\mathbb{R}^m$, if $A \in \mathbb{R}^{m\times n}$

$\{u, v\}$ L.Ind $\Leftrightarrow$ $\{u+v, u-v\}$ L.Ind. ✓ $au+bv=0$

$\frac{a+b}{2}(u+v) + \frac{a-b}{2}(u-v) = 0$

L.Ind

1. solutions of nonhomogeneous systems $\grave{A}x=b$.

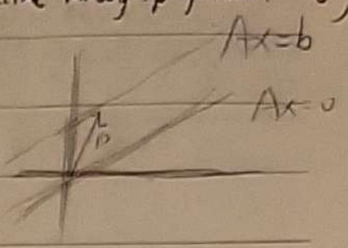
eg $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-1\\2\\0\end{bmatrix}+x_3\begin{bmatrix}1\\0\\1\end{bmatrix}=\vec{p}+x_3\vec{v}=\vec{p}+t\vec{v}\ (t\in\mathbb{R}^1)$ ← scalar

a point

(line through $\vec{p}$ parallel to $\vec{v}$)

2. Solutions of homogeneous systems $Ax=0$

eg. $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\0\\1\end{bmatrix}=x_3\vec{v}=t\vec{v}\ (t\in\mathbb{R}^1)$



3. to $Ax=b$ 对一些给定b，一个solution为 $\vec{p}$ (that is, $Ap=b$)，则 solution set 为 $\vec{w}=\vec{p}+t\vec{v}$，$\vec{v}$ 为 $Ax=0$ 的 ∀ solution

or: if $Ax=b$ has a solution. solution is unique precisely when $Ax=0$ only has trival solution

对 Vector Space V, $S=\{\vec{v}_1,\dots,\vec{v}_p\}\subseteq V$; $x_1\dots x_p$ scalar (in F)

4. Linear Independent $x_1\vec{v}_1+x_2\vec{v}_2+\dots+x_p\vec{v}_p=\vec{0}$ only trival solution (令 $S=\phi$ L.Ind)

Linear dependent $x_1\dots x_p$ not all zero. 无free var = 每row 有 pivot pos.

\#1 (columns of A are L.Ind iff $Ax=0$ has only trival solution)

\#2 $S=\{\vec{v}_1\dots\vec{v}_p\}$ L.dep iff at least one of vectors is L.Comb of others

逆否：L.Ind ⟺ 每一个vector不可由其余L.Comb □ 对非0 x_i，$\vec{v}_i=\frac{x_1}{x_i}\vec{v}_1+\dots$

可推出：若 L.Ind $S\subseteq$ V.S. V, let $v\in V-S$, $S\cup\{v\}$ L.dep ⟺ $v\in span(S)$

(□ ⇒ $a_1s_1+\dots+a_ns_n+av=0$, $a\neq0$ 因 S L.Ind. ∴ v 可表成 $\{s_1,\dots,s_n\}$ L.Comb, $v\in span(S)$

⇐ $v=a_1s_1+\dots+a_ns_n$, note $a_1s_1+\dots+a_ns_n+(-1)v=0$, $-1\neq0$, ∴ $S\cup\{v\}$ L.dep.)

一些注意点见国例6-8

注意见 3. 例1

\#3 any set $\{\vec{v}_1\dots\vec{v}_p\}$ in $\mathbb{R}^n$ is L.dep if $p>n$ 【eg. 4×5 matrix 的col L.dep】

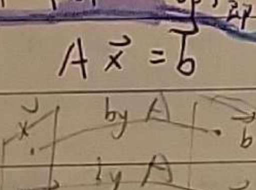
p var 但 n eq must has free var

\#5 If $\{\vec{v}_1\dots\vec{v}_p\}$ in $\mathbb{R}^n$ contains zero vector (0 vec=0) then L.dep □ 有 $\vec{v}_k=0$ 则 $5\vec{v}_k=0$ (5≠0)

\#6 If $\{\vec{v}_1\dots\vec{v}_p\}$ is L.dep, then $\{\vec{v}_1\dots\vec{v}_p,\vec{v}_{p+1}\}$ also L.dep; (逆否) $\{v_1\dots v_p,v_{p+1}\}$ L.Ind, then $\{v_1\dots v_p\}$ L.Ind

即 $S_1\subseteq S_2\subseteq V$, S_1 L.dep ⇒ S_2 L.dep; S_2 L.Ind ⇒ S_1 L.Ind □ 不全0 $a_1\dots a_p$, $a_1\vec{v}_1+\dots+a_p\vec{v}_p+0\vec{v}_{p+1}=0$

5. $A\vec{x}=\vec{b}$ 和 $A\vec{u}=\vec{0}$, $A_{m\times n}$



Transformation/mapping T from $\mathbb{R}^n$ to $\mathbb{R}^m$, $T:\mathbb{R}^n\to\mathbb{R}^m$

each $\vec{x}$ in $\mathbb{R}^n$ to $T(\vec{x})$ in $\mathbb{R}^m$; $\mathbb{R}^n$: domain of T, $\mathbb{R}^m$: codomain of T

$T(\vec{x})$ in $\mathbb{R}^m$ is image of $\vec{x}$, set of all images in range of T

vector space homomorphism, 保证 V, W 有同代数结构 $ax+by\to aT(x)+bT(y)$ 仍线性

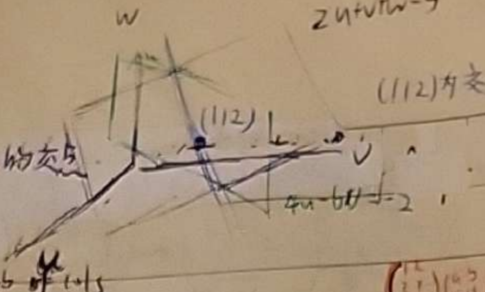
T线性可视为 $\vec{u}$ 和 $\vec{v}$ 和 之后是 变换与不变换 的同时满足，泛化 $T:V\to W$

6. T is linear if i) $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$ for all $\vec{u},\vec{v}$ in domain

ii) $T(c\vec{u})=cT(\vec{u})$ for all scalars c and for all $\vec{u}$ in domain

($T(\vec{0})=\vec{0}$ 有用结论

$T(c\vec{u}+d\vec{v})=cT(\vec{u})+dT(\vec{v})$)



(112)为交点, 即sol

例1 对方程组 $\begin{cases}2u+v+w=5\\4u-6v=-2\\-2u+7v+2w=9\end{cases}$ 以行来看是plane的交点

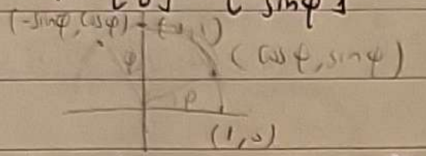
但以列来看, $u\begin{bmatrix}2\\4\\-2\end{bmatrix}+v\begin{bmatrix}1\\-6\\7\end{bmatrix}+w\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}$, comb 求 1,1,2

$\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}(a\ b)+\begin{pmatrix}2\\4\end{pmatrix}(c\ d)$

证1. $A_{mk}B_{kn}=\sum_{i=1}^{k} a_i^{col}\otimes b_i^{row}$ □ $LHS_{ab}=\sum_{i}^{r} a_{ai}b_{ib}$ $RHS_{ab}=\sum_{i=1}^{k} a_{ai}b_{ib}$ 而 $v\otimes u=vu^T$

例2. $T:\mathbb{R}^2\to\mathbb{R}^2$, rotate each point in $\mathbb{R}^2$ about origin through an angle φ. (逆时针 见图例2)

□ $\begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}\cos\varphi\\\sin\varphi\end{bmatrix}$ $\begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}-\sin\varphi\\\cos\varphi\end{bmatrix}$ $A=\begin{bmatrix}\cos\varphi&-\sin\varphi\\\sin\varphi&\cos\varphi\end{bmatrix}$



例3. Linear T $\mathbb{R}^n\to\mathbb{R}^m$ ($T: m\times n$). i) if $\mathbb{R}^n$ onto $\mathbb{R}^m$ ii) if one to one, ~~relation~~ relation of m & n

□ i) 即each row都有pivot position, $n\geq m$; ii) col L.Ind, $m\geq n$ (事实上, 泛化结果见~~xxx~~ 即P_1)

例4. $\left[\begin{array}{cc|c}1&h&-3\\-2&4&6\end{array}\right]$ h 为何 consistent

□ $\left[\begin{array}{cc|c}1&h&-3\\0&4+2h&0\end{array}\right]$ h ∀. (如$h\neq-2$, x可取0)

$\left[\begin{array}{cc|c}1&h&2\\4&8&k\end{array}\right]$ h k 是多少 i) no sol ii) unique iii) infinite many

□ $\left[\begin{array}{cc|c}1&h&2\\0&8-4h&k-8\end{array}\right]$ i) $8-4h=0$, $k-8\neq0$ ii) $8-4h\neq0$ (与k无关) iii) $8-4h=0$, $k-8=0$

例9. 对 $T_1(a_1, a_2, a_3)=(a_1+a_2+1, a_2-a_3, 2a_3-3)$

□ $T_1(0)=(1,0,-3)\neq0$, 不为L.T

例5. $A_{2\times3}$, one free variable / two free variable $A\vec{x}=\vec{0}$ is a line in $\mathbb{R}^3$ / a plane in $\mathbb{R}^3$ 和之前的两点, 沿伸出线 ($\vec{v_3}=v_1+t(v_2-v_1)$)

例6. if $\vec{w}$ is L.Comb of $\vec{u}$ & $\vec{v}$ in $\mathbb{R}^n$, then $\vec{u}$ is L.Comb of $\vec{v}$ and $\vec{w}$. X

$\vec{w}=0\vec{u}+1\vec{v}$

例7. if none of vectors in $\{\vec{v_1}\vec{v_2}\vec{v_3}\}$ in $\mathbb{R}^3$ is a multiple of other vector, $\{\vec{v_1}\vec{v_2}\vec{v_3}\}$ L.Ind X

{(100)(010)(110)}, 判断三个的不可用互相multiple

例8. $\vec{v_1}$ $\vec{v_2}$ in $\mathbb{R}^4$, $\vec{v_2}$ not a scalar multiple of $\vec{v_1}$, then $\{\vec{v_1},\vec{v_2}\}$ L.Ind. X

$\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\}$ 有zero vec. L.dep

需两者都不是对方scalar multiple

即, linear $T\ \mathbb{R}^n \to \mathbb{R}^m$ determined by $\underline{n \times m}$ identity matrix　　linear Map

即, $\forall$ linear T is matrix T

1. $\mathcal{P}_1$ $T\ \mathbb{R}^n \to \mathbb{R}^m$ linear iff $\exists$ $\underline{\text{unique}}$ matrix $\underline{A}$ s.t　（全J表示见※※2.2和四5Bii）　※※10 $\mathcal{T}_3$　~~这叫作~~ standard matrix

$$T(\vec{x}) = A\vec{x} \quad , \forall \vec{x} \in \mathbb{R}^n$$

(e_j col vector, row $j = 1$)

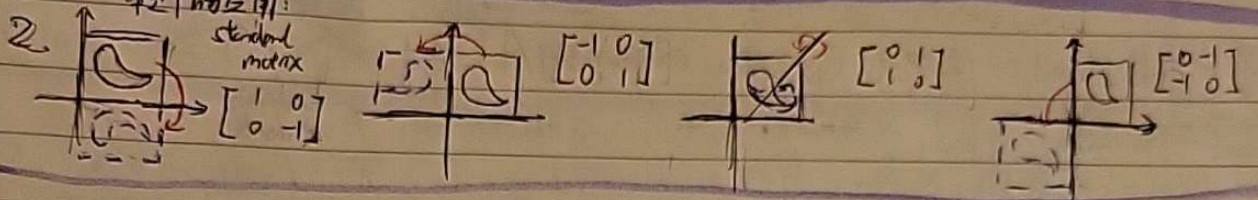
In fact, $A = [T(\vec{e}_1) \ldots T(\vec{e}_n)]$ (A $m \times n$)

☐ $\Rightarrow$) $\vec{x} = I_n\vec{x} = [\vec{e}_1, \ldots, \vec{e}_n]\vec{x}$, $T(\vec{x}) = T([\vec{e}_1, \ldots, \vec{e}_n]\vec{x}) = x_1T(\vec{e}_1) + \ldots + x_nT(\vec{e}_n) = [\underbrace{T(\vec{e}_1), \ldots, T(\vec{e}_n)}_{}]\underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}}$　$\in \mathbb{R}^m$ A

$\Leftarrow$) 回.例6

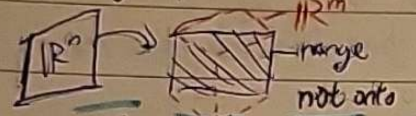
本题1的应用:

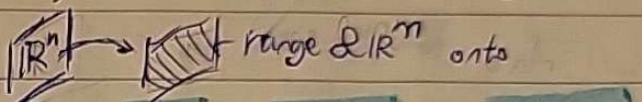
2.



3. $T: \mathbb{R}^n \to \mathbb{R}^m$ onto $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is image of at least one $\vec{x}$ in $\mathbb{R}^n$

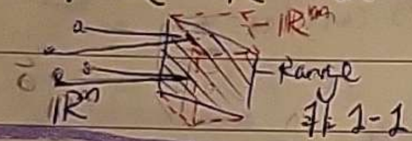
(range of T is codomain $\mathbb{R}^m$) ($\forall \vec{b}$, $T(\vec{x}) = \vec{b}$ exist at least one sol)

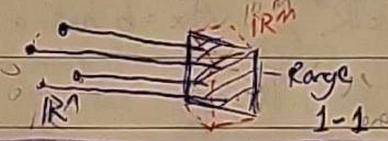




$T: \mathbb{R}^n \to \mathbb{R}^m$ one-to-one if each $\vec{b}$ in $\mathbb{R}^m$ is image of at most one $\vec{x}$ in $\mathbb{R}^n$

($\forall \vec{b}$, $T(\vec{x}) = \vec{b}$ 无解/唯一解)





4. $\mathcal{T}_1$ $T: \mathbb{R}^n \to \mathbb{R}^m$ linear, T one to one iff $\underline{T(\vec{x}) = \vec{0} \text{ trivial sol only}}$.　　即 nullity = 0. 回2.8c4 ~~泛化: $T: V \to W$ linear, T 1-1 $\Leftrightarrow$ Null$(T) = \{0\}$~~

☐ $\Rightarrow$) linear, $T(\vec{0}) = \vec{0}$ (回.6有用结论) $\overset{\text{对 } T(x)=T(0),\ x=0\ (1\text{-}1)}{}$ $\Leftarrow$) let T not 1-1, let $T(\vec{u}) = T(\vec{v}) = \vec{b}$,

$T(\vec{0}) \neq T(\underline{\vec{u} - \vec{v}}) \overset{\text{linear}}{=} T(\vec{u}) - T(\vec{v}) = \vec{0}$, 矛盾.

$\mathcal{T}_2$ $T\ \mathbb{R}^n \to \mathbb{R}^m$ linear, A 为 standard matrix for T. 以 rank/nullity 表示见 四.4

i) T onto $\mathbb{R}^m$ iff columns of A span $\mathbb{R}^m$ ☐ 由定义, 四 $\mathcal{T}$

ii) T one to one iff columns of A L. Ind. ☐ 由定义 L. dep 有 free variable, 无数解

· 即 $AB = [\underset{\text{col}}{\underline{A\vec{b}_1}} \ldots A\vec{b}_n] = \begin{bmatrix} \overset{\text{row}}{\vec{a}_1^T B} \\ \vdots \\ \vec{a}_n^T B \end{bmatrix}$, 也 $= \sum_{i=1}^{k} \underset{\text{col}}{\underline{a_i}}\ \underset{\text{row}}{\underline{\vec{b}_i^T}}$ (定理2)　sum over outer product　· 而 $Ax = \sum \vec{a}_i x_i$

5. $\mathcal{T}_3$ each column of $A_{mk}B_{kn}$ is a linear combination of columns of A using weights from corresponding column of B. (更简证明见 四.3)

☐ $B = [\vec{b}_1, \ldots, \vec{b}_n]$, $A \cdot B = [A\vec{b}_1, \ldots, A\vec{b}_n]$ (拿出 B 的 col vector 使用. $A\vec{b}_i = [\vec{a}_1, \ldots, \vec{a}_n]\vec{b}_i = \sum_{j=1} a_j b_j$)

6. $AB \neq BA$ in general;　$\underline{AB = AC}$ 不一定 $B = C$; $AB = 0$ 不一定 $A = 0$ or $B = 0$
除非 A 可逆, $A^{-1}AB = A^{-1}AC$

$\underset{\text{单位矩阵}}{\underline{I_m}} A = A = AI_n$,　$\underline{A(BC) = (AB)C}$　$A(B+C) = AB + AC$　证明见 四.2.

$(B+C)A = BA + CA$

从 transformation 角度看, 都是先 C 再 B 再 A, 可不必证明, ABC 都是 standard matrix

Date. /

例1. $(AB)^T = (B^T A^T)$

□ $(AB)^T_{ij} = (AB)_{ji} = \sum_k A_{jk}B_{ki}$，而 $(B^TA^T)_{ij} = \sum_k B^T_{ik}A^T_{kj} = \sum_k A_{jk}B_{ki}$

例2. AB n×n, AB invertible, 则 A, B invertible.

□ let $C = (AB)^{-1}$, consider BC, $A(BC) = (AB)C = I$ 同理 CA 为 B^{-1} 若 AB 非square 不对

而令 $BCA = X$, $CAB = B^{-1}XB$, $I = B^{-1}XB$, $X = BIB^{-1} = I$ $\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} = I$ 可逆

例3 AB 的 last col 全为 0, B 无 zero column, 则 A $\overset{\text{cols}}{}$ L.Ind. ✗

□ $AB = [A\overset{B\text{ col}}{\vec{b}_1} \cdots A\vec{b}_n]$, 即 $A\vec{b}_n = \vec{0}$, $\vec{b}_n \neq \vec{0}$, ∴ 有 nontrivial sol, A col L.dep

OR: $(AB)_{:,-1} = 0 = \overset{\text{四}.1}{A}(B)_{:,-1}$ $= \text{四}.3)$ 即 $\sum_{i=1} t_i A_{:,i}$, where $B_{:,-1} = \begin{pmatrix}t_1\\ \vdots\\ t_n\end{pmatrix}$ 而 $t_i \neq 0$, ∴ $\{A_{:,i}\}$ L.dep

B 的 cols, L.dep 则 AB 的 col 也 L.dep ✓

□ 令 $(AB)_{:,i} = A(B)_{:,i}$ 而 $B_{:,i}$ 已 L.dep. coeff 带给 $\{(AB)_{:,i}\}$ 也 = 0

例6. i) $CA = I_n$, then $Ax = 0$ only trivial sol. A can't have more col than row.

□ $Ax=0$, $CAx = 0 = x$, ∴ trivial ~~列<row 否则不会有 trivial sol~~

nullity = 0

ii) $AD = I_m$, then $\forall b \in \mathbb{R}^m$, $Ax = b$ has a sol, A can't have more row than col

□ $A\underbrace{Db}_{x} = b$ ~~列多于row 否则不行 往 每 row 都有 pivot pos.~~

A col span $\mathbb{R}^m$

iii) A as m×n, ∃ n×m C and D s.t. $CA = I_n$ and $AD = I_m$, prove m=n, C=D

□ 可逆 即 同时 满足 i) & ii) — $(C_{nm}(A_{mn})D_{nm}) = D = C$ ∴ 由 i) & ii) m=n

说明 当 A 可逆, $Ax=0$ only trivial (nullity = 0), A span $\mathbb{R}^n$ (rk = n), 满足 $\boxed{\text{三}}.3$.

例7. $A \leftarrow A + \varepsilon B$, ε small 证 $A^{-1} \leftarrow A^{-1} - \varepsilon A^{-1}BA^{-1}$ □ $A^{-1} = (A+\varepsilon B)^{-1} = [A(I+\varepsilon A^{-1}B)]^{-1} = (I+\varepsilon A^{-1}B)^{-1}A^{-1}$

Optim P10.3 $= (I - \varepsilon A^{-1}B + \varepsilon^2(A^{-1}B)^2 - \dots)A^{-1} \approx A^{-1} - \varepsilon A^{-1}BA^{-1}$, 即 ignore negligible $\varepsilon^2(A^{-1}B)^2 - \dots$ term

例4 A invertible n×n, $B_{n\times p}$, 证 $[A\ B] \sim [I\ X]$, $X = A^{-1}B$

□ $(\underbrace{E_p \cdots E_1}_{A^{-1}})A = I$, ∴ $(E_p \cdots E_1)B = X = A^{-1}B$

例6. $A = -A^T$, 则 diag = 0, 且 $\forall x$, $x^TAx = 0$ □ 1° $A + A^T = 0$

2° $(x^TAx)^T = x^TAx = -x^TAx$ ∴ $2x^TAx = 0$

例5 A B X n×n, AX, A−AX 可逆, $(A-AX)^{-1} = X^{-1}B$, 证 B 可逆, 并 solve X

NOTE 除了证 $B^{-1}B = BB^{-1} = I$ 以外 (这里此法较难), 用 $B = X(A-AX)^{-1}$. 可逆 乘积 仍可逆 台P2)

□ $X = B(A-AX)$, $X = (I-BA)^{-1}BA$

注1. AB−I 可逆 ⇒ BA−I 可逆 □ WTS BA−I 不可逆 ⇒ AB−I 不可逆 四.4.2 ∴ BA−I evl = 0

∴ BA evl = 1 四.6(5)

四P6 ∴ AB 有 evl = 1 AB−I evl = 0 不可逆

注 I−AB 可逆 ⇒ I−BA 可逆 法1 $-(I-AB) = AB-I$, apply 1° 法2 若 $(AB)^k \to 0$ 时

$(I-AB)^{-1}$ ∃, $(I-AB)^{-1} = I + AB + ABAB + ABABAB + \dots = I + A(I + BA + BABA + \dots)B = I + A(I-BA)^{-1}B$

optim P10.3

symmetric: $A=A^T$ (对∀square A, $A+A^T$ symmetric)

skew-symmetric: $A^T=-A$ ($A-A^T$ skew-symmetric)

Invertible & Transpose

$W=[X\ \vec{x}_0]$ $W^T=\begin{bmatrix}X^T\\ \vec{x}_0^{\,T}\end{bmatrix}$

$\begin{bmatrix}A&B\\0&D\end{bmatrix}^T=\begin{bmatrix}A^T&0\\B^T&D^T\end{bmatrix}$

$(rA+sB)_{ij}=rA_{ij}+sB_{ij}$

$(rA^T+sB^T)_{ij}=rA_{ji}+sB_{ji}$

Date /

1. Transpose $A_{m\times n}\to A^T_{n\times m}$

$\begin{bmatrix}a&b\\c&d\end{bmatrix}\xrightarrow{T}\begin{bmatrix}a&c\\b&d\end{bmatrix}$ $\begin{bmatrix}-5&2\\1&-3\\0&4\end{bmatrix}\xrightarrow{T}\begin{bmatrix}-5&1&0\\2&-3&4\end{bmatrix}$ $\begin{bmatrix}1&1&1&1\\-3&5&-2&7\end{bmatrix}\xrightarrow{T}\begin{bmatrix}1&-3\\1&5\\1&-2\\1&7\end{bmatrix}$

'matrix' undo!

$(A^T)^T=A$ (idempotent)

$(rA+sB)^T=rA^T+sB^T$, $r,s\in F$ (linearity)

$\det(A)^T=\det A$ (后图.1)

$(AB)^T=B^TA^T$ (e 查例1)

求逆了想成线性变换后再变回去(如对$[\,]$$[\,]$)

2. $A_{n\times n}$ invertible if $\exists!\,C_{n\times n}$ s.t. $CA=I_{n\times n}=AC$, C即A^{-1}

如C不唯一, $B=BI=B(AC)=(BA)C=IC=C$ ∴ $B=C$

定义上因两边都要满足, A必为方阵, 否则m×n表达有误

3. 对于 2×2 的 $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ if $|A|=ad-bc\neq0$, $A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

日后推解方程

if $|A|=0$, not invertible ($A\sim\begin{bmatrix}a&b\\0&d-\frac{b}{a}c\end{bmatrix}$ 的$|A|=0$, $[0,0]$, 不可逆)

4. 若 $A_{n\times n}$ invertible $\Leftrightarrow\forall\vec b\in\mathbb{R}^n$, $A\vec x=\vec b$ has unique sol $\vec x=A^{-1}\vec b$

见 P77.1

□ 若不uniq. 设$\vec u\neq\vec x$ 有 $A\vec u=\vec b$, $A^{-1}A\vec u=\vec u=A^{-1}\vec b=\vec x$

如是逆, $A_{n\times n}$可逆$\Leftrightarrow$ col of A L.Ind, span $\mathbb{R}^n$

$[A\ b]\sim[I\ x]$ 可取之, 见老例4

5. 1. $(A^{-1})^{-1}=A$, i.e. A is invertible, then A^{-1} invertible □ $AA^{-1}=I=A^{-1}A$

2. $(AB)^{-1}=B^{-1}A^{-1}$ □ $AB\cdot(B^{-1}A^{-1})=AIA^{-1}=I$, 同理 $(B^{-1}A^{-1})AB=I$ 可逆乘积可逆

回例1

3. $(A^T)^{-1}=(A^{-1})^T$ □ $(A^{-1})^TA^T=(AA^{-1})^T=I^T=I$ 4. $r\neq0$, $(rA)^{-1}=r^{-1}A^{-1}$ 可逆

6. 对 $A_{m\times n}$, $B_{m\times p}$, 考虑 Augment matrix $m\times(n+p)$ $(A|B)$; note: 对$\forall M_{m\times n}$, $M(A|B)=(MA|MB)$

□ 对 $M(A|B)$ 的前n列, $M(A|B)_{ij}$, $j=1..n$, 为 $\sum_k M_{ik}A_{kj}=MA_{ij}$, 同理 $j=n+1..n+p$, $M(A|B)_{ij}=MB_{ij}$

以此用于找inverse. $A_{n\times n}$可逆 $\Leftrightarrow(A|I_n)$ elementary row operation $\sim(I_n|A^{-1})$ (教材老例4)

只对row, row才是本

按6. 回 β(iii)

□ $\Rightarrow$ 对 $A^{-1}(A|I_n)=(I_n|A^{-1})$, 因 $A^{-1}=E_p\cdots E_1$, ∴ $E_p\cdots E_1(A|I_n)=(I_n|A^{-1})$ 为 elementary matrix 代表 row operation

$\Leftarrow$ 令 M = row op 的乘积, 即 $M=E_p\cdots E_1$, $\overset{M}{E_p\cdots E_1}(A|I_n)=(I_n|B)=(MA|MI_n)$ ∴ $\begin{cases}MA=I_n\\M=B\end{cases}$

WTS $B=A^{-1}$. 但由 $MA=I_n$, $M=A^{-1}=B$

7. Linear transform 为 special case of affine transform (linear + translation): $f(x)=Ax+b$

Def: T affine iff ∀scalar λ $f(\lambda x+(1-\lambda)y)=\lambda f(x)+(1-\lambda)f(y)$

7. 因

$X = A_{11}^{-1} A_{21} \quad Y = A_{11}^{-1} A_{12}$

应用.

④ If A_{11} 可逆, $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$ 令 $S = A_{22} - A_{21} A_{11}^{-1} A_{12}$

同理 A_{22} 可逆, 舒尔逆为 $A_{11} - A_{12} A_{22}^{-1} A_{21}$, S 称为 schur complement

to $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ 与 A_{11} 都可逆, S 也可逆.

□ lemma $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ 可逆 iff B, C 都可逆, (B C, square)

□ (⇒) $\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ 得 $\begin{cases} BX = I \\ WC = I \end{cases}$; (⇐) $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

$\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} = I \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} I$, 左边为三个可逆矩阵, 相乘必可逆 (见上页), 因此 $\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}$ 可逆, 由 lemma, S 可逆.

应用. a standard set of differential equations is transformed by Laplace transforms into $\begin{bmatrix} A - sI_n & B \\ C & I_m \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{u} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{y} \end{bmatrix}$, where $A_{n\times n}$, $B_{n\times m}$, $C_{m\times n}$, s a variable, $\vec{u}$, $\vec{x}$, $\vec{y}$ function of s, $\underset{\uparrow \text{input}}{\vec{u}}$ in $\mathbb{R}_m$ ($\vec{u}_{m\times 1}$), $\vec{x}$ in $\mathbb{R}^n$ (state vector), $\vec{y}$ in $\mathbb{R}^m$ (output)

有 transfer function $W(s)$ s.t. $W(s)\vec{u} = \vec{y}$ (input ↦ output). 如 $A - sI_n$ 可逆, $W(s)$ 为舒尔补 逆

□ $\begin{cases} (A - sI_n)\vec{x} + B\vec{u} = \vec{0} \\ C\vec{x} + I_m\vec{u} = \vec{y} \end{cases} \Rightarrow \begin{aligned} &\vec{x} = (A - sI_n)^{-1}(-B\vec{u}) \\ &\Downarrow \\ &\vec{y} = (I_m - C(A - sI_n)^{-1}B)\vec{u} \end{aligned} \Rightarrow$ 即 $W(s)$ 为 $\begin{bmatrix} A - sI_n & B \\ C & I_m \end{bmatrix}$ 中 $(A - sI_n)$ 的 schur complement

由上得 舒尔补逆 有可逆情况, 则可找出 $W(s)^{-1}$, output ↦ input

例2. 如 $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{逆时针}\ \theta_1\ \text{rotate}} v_1$ 证 $\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$

$\xrightarrow{\text{顺时针}\ \theta_2} v_2$

□ $v_1 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}$, $v_2 = \begin{pmatrix} \cos-\theta_2 & -\sin-\theta_2 \\ \sin-\theta_2 & \cos-\theta_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta_2 \\ -\sin\theta_2 \end{pmatrix}$

$\cos(\theta_1 + \theta_2) = \cos(\text{ang between } v_1\ v_2) = \dfrac{\langle v_1\ v_2 \rangle}{\|v_1\| \cdot \|v_2\|}$

例1: $A_{m\times n}$ 的 col form a basis of $\mathbb{R}^m$, 则 m=n

□ i) col span $\mathbb{R}^m$, $n \geq m$ (回.练116 ii)) ii) col L.ind (回.J #3) $n \leq m$ ⇒ n=m

if $A_{m\times m}$, Col A ≠ $\mathbb{R}^m$, 则 Nul A ≠ {0}, A 不可逆

□ 由回.3, rkA≠n ∴ nullity≠0, 方阵非满秩, 不可逆 (P4.1.) OR 即非∀ row有pivot pos, ∴ Ax=0 有 nontrivial sol

$A_{3\times5}$ has 3 pivot col, Is Col A = $\mathbb{R}^3$? Is Nul A = $\mathbb{R}^2$?

□ Col A = $\mathbb{R}^3$, 但 nullspace ≤ $\mathbb{R}^5$ (dim N(A) ≠ N(A))

$A_{4\times7}$ has 3 pivot col, Is Col A = $\mathbb{R}^3$? What is dim Nul A?

□ A 4 row Col A ≤ $\mathbb{R}^4$ nullity = 7 − rk A = 7−3 = 4

Each line in $\mathbb{R}^n$ is one-dimensional subspace of $\mathbb{R}^n$. ✗

□ ~~line 若经 origin 才是 subspace, dim only def for subspace~~

回 $A_{6\times4}$, $B_{4\times6}$, AB 不可逆 ✓ □ B col > row, col B L.dep (回 J #3). ∴ Bx=0 有 nontrivial sol, 两边同乘 A

ABx = A0 = 0, ∴ AB nullity≠0, 不 full rk

Invertible 2
VS 5

Date. /

1. $A_{n\times n}$. 以下同真同假

那不需同时满足 $CA=AC=I$，因 square

a. A is invertible ; A^T is invertible ; there is $n\times n$ C s.t $CA=I$; there is $n\times n$ D st. $AD=I$

b. A row equivalent to $n\times n$ Identity matrix

c. A has n pivot position ; cols of A span $\mathbb{R}^n$; $\forall \vec{b}\in\mathbb{R}^n$, $A\vec{x}=\vec{b}$ has one unique sol

只能是 unique sol. 因 $\forall b$, 不会出现 [0 ... 1]

d. $A\vec{x}=\vec{0}$ has only trival sol ; cols of A L. Ind.

f. linear T $\vec{x}\mapsto A\vec{x}$ is one-to-one ; maps $\mathbb{R}^n$ onto $\mathbb{R}^n$; T bijective ; T invertible

导出结果 III.4

2、let $T:\mathbb{R}^n\to\mathbb{R}^n$ be L.T. and let A be standard matrix for T. 即 relate L_A & A，见 17

T is invertible iff A is invertible. A^{-1} would be standard matrix for T^{-1}

3. If $A_{m\times n}$, $B_{n\times o}$, A partition into x rols and y cols, B partition into y rows and z rows

then they are comfortable for block multiplication

eg. $AB=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}B_1\\ B_2\end{bmatrix}=\begin{bmatrix}A_{11}B_1+A_{12}B_2\\ A_{21}B_1+A_{22}B_2\end{bmatrix}$ (i,j) in $col_k(A)\cdot row_k(B)$ 为 $a_{ik}b_{kj}$ thus $\Sigma(i,j)$ 为 (i,j) in AB

In general $AB=[col_1\ col_2 \dots col_n]\begin{bmatrix}row_1\\ row_2\\ row_n\end{bmatrix}=col_1row_1+\dots+col_nrow_n$

4. block upper triangular $A=\begin{bmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix}$, assume $A_{11\,p\times p}$, $A_{22\,q\times q}$, A ~~invertible~~ invertible

证 $A^{-1}=\begin{bmatrix}A_{11}^{-1} & A_{11}^{-1}A_{12}A_{22}^{-1}\\ 0 & A_{22}^{-1}\end{bmatrix}$

口 设 $AB=I_{p+q}$ — $\begin{bmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix}\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}=\begin{bmatrix}I_p & 0\\ 0 & I_q\end{bmatrix}$

$$\begin{cases}A_{11}B_{11}+A_{12}B_{21}=I_p\\ A_{11}B_{12}+A_{12}B_{22}=0\\ A_{22}B_{21}=0\\ A_{22}B_{22}=I_q\end{cases}\Rightarrow$$

- A_{22} 方阵，令 $B_{22}=A_{22}^{-1}$
- $B_{21}=A_{22}^{-1}\cdot 0=0$
- $A_{11}B_{11}=I_p \therefore B_{11}=A_{11}^{-1}$
- $B_{12}=-A_{11}^{-1}A_{12}A_{22}^{-1}$

见 P10.1

vector space 用这三点可推出 subspace 为 vector space，H 只是 $\mathbb{R}^n$ 的一部分，但保留了代数结构

5. Subspace of $\mathbb{R}^n$ is anyset H, a) zero vector in H b) each $\vec{u},\vec{v}$ in H, $\vec{u}+\vec{v}$ in H

可被 c) 推出：$k=0$, $k\vec{u}=\vec{0}\in H$ 但仍需要列出，见 P10.1 NOTE

c) each $\vec{u}$ in H, k scalar, $k\vec{u}$ in H. 一条不过原点的线不是 subspace

$A\vec{x}=\vec{b}$ 无解就是说 b 不在 A 的列空间(一个子空间)

$\vec{v}_1\dots\vec{v}_p$ in $\mathbb{R}^n$, All L. Cons of $\vec{v}_1\dots\vec{v}_p$ is subspace in $\mathbb{R}^n$; column space of $A_{m\times n}$ is subspace in $\mathbb{R}^m$

(The null space of $A_{m\times n}$ is set of solutions of homogeneous equation $A\vec{x}=\vec{0}$, a subspace in $\mathbb{R}^n$

注意 col space 为 $\mathbb{R}_m$，为 $\vec{b}$ 的 dim. ∴ col space 也用所有 output $\vec{b}$ 的集合 (只有 $\mathbb{R}^n$)

$\vec{b}$ 一直在 col space，一方面因 $\vec{b}$ 是 subspace. 因 A 的 col 即是线性组合基 basis 所在的向量 张成的空间即 col space 9

一方面 $\vec{b}$ 永远可被线性连接得到. 在被压缩的情况下，null space 即被可连接到 $\vec{b}$ 的向量张成.

$H_1 = \{(a-3b, b-a, a, b) : a, b \in \mathbb{R}\}$ H_1 is subspace of $\mathbb{R}^4$ $\begin{bmatrix} b-a \\ 0 \\ b \end{bmatrix} = a\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec v_1} + b\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec v_2}$

$H_2 = \{(a,b,c,d) : a-2b+5c = d$ and $c-a=b\}$, H_2 is subspace of $\mathbb{R}^4$ (P 7.4三)

Date. / $\begin{cases} a-2b+5c-d=0 \\ -a-b+c=0 \end{cases}$ set of homogeneous equations, H_2 subspace

例1 $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$ find basis for null space of A.

□ 考虑 $Ax=0$, $[A|0] = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\vec x = \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{u} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{v} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{w}$ span{u, v, w} = Nul(A) 且 u v w L.Ind ← basis {u, v, w}

Note, 上述 =0 只有 x_2, x_4, x_5 都为0

证1. Col A ∩ Nul A^T = $\{\vec 0\}$ □ 对任 $Au \in$ Col A, 若 $Au \in$ Nul A^T, $A^TAu = 0$ 即 $u \in N(A^TA) = N(A)$ $\boxed{四}$.5
∴ $Au = 0$ 即 intersection 只有0

例2 find basis for col space of $A = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (pivot column 为 basis)

□ $\forall \vec v \in \mathbb{R}^4$ $\vec v = \sum^5 c_i \vec a_i$, 但 $\vec a_3, \vec a_4$ 可被 l. comb. ∴ L.Ind basis $\{a_1, a_2, a_5\}$

例3 $\vec v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ $\vec v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\vec x = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, $\beta = \{\vec v_1, \vec v_2\}$, then β is basis for $H = \mathrm{Span}\{\vec v_1, \vec v_2\}$ since $\vec v_1 \vec v_2$ L.Ind.

If $\vec x$ in H, find coordinate vector □ 因 $\vec x \in H$, $\vec x = \sum^2 x_i \vec v_i$ $\left[\begin{array}{cc|c} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{cases} x_1 = 2 \\ x_2 = 3 \end{cases}$

所以们有 $[x]_{\text{natural}} = Q_{\beta \to \text{natural}} [x]_\beta$

$\begin{bmatrix} 3 & -1 \\ 6 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$

例4 $\begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}$ 简化为 $\begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 可知 第3、4列为 free variable, 第1、2、5列为 basis, 但只有 $\begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ -8 \\ 7 \\ 11 \end{bmatrix}$ 是而不是 $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ 和 $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. 即化简后的不是 basis

pivot col 表示线性相关性, 各 pivot col 间 L.Ind. 有 free variable 的即可被他们表示

例5 $\alpha_1 \ldots \alpha_r$ L.Ind. $u \in$ span A, $v \notin$ span A, 证 $\forall t \in F$, $\{tu+v, \alpha_1 \ldots \alpha_r\}$ L.Ind

□ $0 = \sum_{i=1}^{r} b_i \alpha_i + b_{r+1}(tu+v)$, 若 $b_{r+1} \neq 0$, $tu+v = t\sum_{i=1}^{r} a_i \alpha_i + v = \sum_{i=1}^{r} \frac{-b_i}{b_{r+1}} \alpha_i$, 则 $v \in$ span A

矛盾; $b_{r+1} = 0$, 而 α_i L.Ind, b_i $i=1,\ldots,r$ $=0$ ∴ L.Ind

若 $k \geq 2+n$ $a_i \in \mathbb{R}^n$, 则 $\{a_1 \ldots a_k\}$ L.dep. 证 ∃ scalar $\overset{\text{至少有1个非0}}{\underline{b_1 \ldots b_k}}$ st. $\sum^k b_i = 0$, $\sum^k b_i a_i = 0$

□ def $\vec{\bar a_i} = \begin{bmatrix} 1 \\ a_i \end{bmatrix} \in \mathbb{R}^{n+1}$, $\bar a_i$ 仍 L.dep, ∴ 有非全0 b_i st. $\sum b_i \bar a_i = \vec 0 = \begin{bmatrix} \sum b_i \\ \vdots \end{bmatrix}$

例6 AB, A diagonal matrix, 即 scalar times rows of B ✓ A row i 为 $[0\ 0 \ldots d_i \ldots 0]$, AB row i 为 $[0\ 0 \ldots dB \ldots 0]$

If $AC = 0$, then $A = 0$ or $C = 0$ ✗ $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$(A+B)(A-B) = A^2 - AB + BA - B^2$, only when A, B commute 才为 $A^2 - B^2$ ✓

每个 square matrix is a product of elementary matrixs. ✗ 即 若 $A = E_p \cdots E_1 I$, 但 A 不一定可逆 $\boxed{23}$ 11)
化 A 为 I

A 可逆, $r \neq 0$, $(rA)^{-1} = r^{-1}A^{-1}$ ✓

$A_{n\times n}$ $\det(cA) = c^n \det(A)$ ✓ P_5 P_{viii})

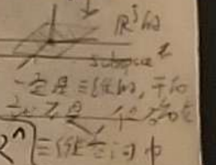
$\boxed{10}$ 一般 $AB \neq BA$, $\det BA = \det AB$ ✓ P_7 P_2, $\det(A+B) = \det A + \det B$

$\begin{bmatrix}3\\8\\-1\end{bmatrix}\begin{bmatrix}6\\2\\5\end{bmatrix}$ 是 L.Ind 但不 span $\mathbb{R}^3$，因此不是 basis

dim 2 & det 3 & span 2

$\mathbb{R}^2$ 不是 $\mathbb{R}^3$ 的 subspace，$\mathbb{R}^2$ 的元素 2个值，不是 $\mathbb{R}^3$ 内包的 subset.

$H=\{\begin{bmatrix}s\\t\\0\end{bmatrix}\}$ 才是 $\mathbb{R}^3$ 的 subspace



一定是三维的，平面 并不是，但在 三维空间中

pivot col of matrix A form a basis for col space of A 见 $P_2.3$

1. Basis for vector space V (P8) is L.I. set in V that spans V.

eg1. $\{\vec{e_1} .. \vec{e_n}\}$ $\boxed{\text{standard basis}}$ for $\mathbb{R}^n$, thus 可逆 n×n 矩阵的 cols 即 basis for $\mathbb{R}^n$ （$\mathcal{C}(A)=\mathbb{R}^n$）

eg2. 对 $V=\{0\}$，因 $\text{span}(\phi)=\{0\}$，ϕ 为 V 的 basis (P_2.定义)

2. if set $\beta=\{\vec{b_1} .. \vec{b_p}\}$ (ordered) is basis for subspace H, for each $\vec{x}$ in H, coordinate of $\vec{x}$ relative to basis β (is unique, 见 J_1) is $\vec{x}=c_1\vec{b_1}+c_2\vec{b_2}+\cdots+c_p\vec{b_p}$, and $[\vec{x}]_\beta=\begin{bmatrix}c_1\\\vdots\\c_p\end{bmatrix}$ is coordinate vector relative to β. （$\vec{x}=[\vec{b_1}..\vec{b_p}][\vec{x}]_\beta$ 线性方程 解唯一 $\Rightarrow [\vec{b_1}..\vec{b_p}]^{-1}\vec{x}=[\vec{x}]_\beta$）

The dimension of a nonzero subspace H is number of vectors in any basis for H!

The rank of A is the dimension of col space of A. } dim of subspace {0} is defined as 0

如 $\vec{x}$ 左 A 变换后 $\vec{b}$ 为直线，rank = 1. $\vec{b}$ 为平面，rank = 2 ..., 即 number of dim in output $\vec{b}$.

3. (Rank Theorem) If A has n col, $\overset{\text{pivot col}}{\text{rank}}A+\overset{\text{free variable}}{\dim}(\text{Nul}(A))=n$:

泛化的 proof: $T: V\to W$ linear. $\overset{\dim(\text{Null}(T))}{\text{nullity}(T)}+\overset{\dim(\text{Range}(T))}{\text{rank}(T)}=\dim(V)$ ※※3 c)

□ let $\dim(V)=n$, $\text{nullity}(T)=k$, $\{v_1,..,v_k\}$ basis for Null(T), 可 extend 为 $\{v_1,..,v_n\}$ 为 V 的 basis, 令 $S=\{T(v_{k+1})..T(v_n)\}$

WTS S 为 Range(T) 的 basis. i) S span: $\text{Range}(T)\overset{※※10\,J_2}{=}\text{span}(T(v_1),..,T(v_n))=\text{span}(T(v_{k+1}),..,T(v_n))=\text{span}(S)$

ii) S L.Ind. let $\sum_{k+1} a_iT(v_i)=\vec{0}$, T linear, $T(\sum_{k+1} a_iv_i)=0$, $\sum_{k+1} a_iv_i\in\text{Null}(T)$, ∴ 可被 $\{v_1..v_k\}$ 表示: $\sum_{k+1} a_iv_i=\sum_1^k b_jv_j$ (※※3. J_1)

由 $\{v_1,..,v_n\}$ V basis, $\sum^k b_iv_i-\sum_{k+1} a_iv_i=0$, $a_i=b_i=0$ ∴ $\{T(v_{k+1}),..T(v_n)\}$ basis, rank $k=n-k$

4. $A_{n\times n}$，A 可逆除了包含上页的定理外：

a) Col of A form basis of $\mathbb{R}^n$

b) Col space $A=\mathbb{R}^n$; dim Col A = n ; rank A = n

□ 以泛化表示: linear $T: V\to W$, T bijection ⟺ $\text{rank}(T)=\dim(T)$; T 1-1 ∴ nullity = 0 (因 J_2) ($A_{n\times n}$ 可记为 linear T (同 ...))

~~若 V W dim 同, 1-1 即 onto! rank T = dim(Range(T)) = dim V = dim W ∴ range(T) = W~~ (※※3 J_3), by def onto

5. Determinant $\det A_{n\times n}=a_{11}\det A_{11}-a_{12}\det A_{12}+\ldots+(-1)^{n+1}a_{1n}A_{1n}=\sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det A_{1j}$ 作为 (1, j) cofactor C_{1j}

（对于三阶矩阵，det 的证明 $\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}\sim\begin{bmatrix}a_{11}&a_{12}&a_{13}\\0&a_{11}a_{22}-a_{12}a_{21}&a_{11}a_{23}-a_{13}a_{21}\\0&0&a_{11}\Delta\end{bmatrix}$ $\Delta=a_{11}a_{22}a_{33}+$ 即 det A. 如 A 可逆, $\Delta\neq0$. 即 Δ 有 重要意义, 称为 行列式）

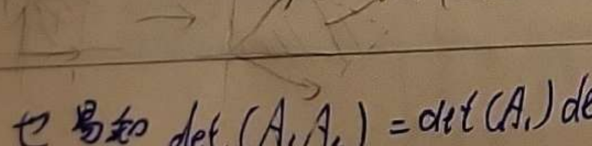
cofactor 不只是 C_{1j}. 可对 ∀ row / col cofactor expansion.

6. Row operations: 1° 两行互换, det 改变号 2° 一行加到另一行, det 不变 3° 一行 ×k, det ×k

推论 1° 两行相同, det = 0. 2° 两行成比例, det = 0 3° 一行都 0, det = 0

从大中知 A 不可逆 ⟺ (无 pivot position) det = 0

7. 行列式 可表示 线性变换后 (1, 0) (0, 1) (basis) 围成的区域 扩大了多少倍，因此 只要 det ≠ 0，



det (二维 det 即面积 ...)，即 化为不能用当前维度表示 (不一定 0!!!)，A 就 可用相反的变换 变回来，A^{-1} 存在

即! 一般 linear T invertible 除非 m个点到 同一点 压缩空间, ∴ 扩大 0 倍 (压没了)

也易知 $\det(A_1A_2)=\det(A_1)\det(A_2)$ 拉伸两次

应用 $A=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ $B=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ Pauli spin matrices, $AB=-BA$, anticommute
量子力学中 electron spin

应用，寻找 (-2,0) (0,3) (1,3) (-1,0) 围成平行四边形面积 平移后为 $\overset{A}{(0,0)}, \overset{B}{(2,3)}\overset{C}{(3,3)}\overset{D}{(1,0)}$
∵ B(2,3) D(1,0) $\left|\begin{bmatrix}2&1\\3&0\end{bmatrix}\right| = |3| = 3$

例 to $A^n=0$ for some $n>1$, $(I-A)^{-1}$?

□ $(I-A)\underbrace{(I+A+A^2+\dots+A^{n-2}+A^{n-1})}_{(I-A)^{-1}} = I - A^n = I$

应用. let a, b positive, find area of $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1$

□ 从单位圆映射到 $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1$ 的方程

$A\vec{u}=\vec{x}$, $\vec{u}=\begin{bmatrix}u_1\\u_2\end{bmatrix}$ $\vec{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}au_1\\bu_2\end{bmatrix}$

得知 $A=\begin{bmatrix}a&0\\0&b\end{bmatrix}$

area $= |\det A| \cdot \underset{}{\overset{\text{原先面积}}{\text{Area}_D}} = ab\pi(1^2) = \pi ab$

例 Vandermonde $V=\begin{bmatrix}1&x_1&x_1^2&\cdots&x_1^{n-1}\\1&x_2&x_2^2&\cdots&x_2^{n-1}\\\vdots&\vdots&\vdots&&\vdots\\1&x_n&x_n^2&\cdots&x_n^{n-1}\end{bmatrix}$. $\vec{y}=(y_1,\dots,y_n)$ in $\mathbb{R}^n$, $\vec{c}=(c_0,\dots,c_{n-1})$ in $\mathbb{R}^n$

$V\vec{c}=\vec{y}$. $p(t)=c_0+c_1t+\dots+c_{n-1}t^{n-1}$

a) show $p(x_1)=y_1$, $p(x_2)=y_2 \dots p(x_n)=y_n$. $p(t)$ interplolating polynominal for $(x_1,y_1)\dots(x_n,y_n)$. 即 graph pass through points

b) 若 $x_1,\dots,x_n$ distinct. show col of V L.Ind.

□ a) $\begin{bmatrix}1&x_1&x_1^2&\cdots&x_1^{n-1}\\1&x_2&x_2^2&\cdots&x_2^{n-1}\\1&x_n&x_n^2&\cdots&x_n^{n-1}\end{bmatrix}\begin{bmatrix}c_0\\c_1\\\vdots\\c_{n-1}\end{bmatrix}=\begin{bmatrix}c_0+c_1x_1+\dots+c_{n-1}x_1^{n-1}\\\vdots\\c_0+c_1x_n+\dots+c_{n-1}x_n^{n-1}\end{bmatrix}=\begin{bmatrix}p(x_1)\\p(x_2)\\\vdots\\p(x_n)\end{bmatrix}=\begin{bmatrix}y_1\\\vdots\\y_n\end{bmatrix}$

b) n-1元多项式最多n-1个解. 若 $\vec{y}=\vec{0}$, $p(x_1)\dots p(x_n)$ n个多项式都为0. 若 L.dep. 则 $(c_0\dots c_{n-1})$ 有nontrivial解.
因 $x_1\dots x_n$ 不同, 即 n-1次多项式有n个解, 矛盾. ∴ 只有 $c_0=c_1=\dots=c_{n-1}=0$ 才使n个多项式为0. $\vec{c}$ 只有 trivial sol. 则此 c L.Ind

应用: 证 $\det\begin{bmatrix}A&0\\C&D\end{bmatrix}=\det A\cdot\det D=\det\begin{bmatrix}A&B\\0&D\end{bmatrix}$ (用于求eigenvalues (三角矩阵))

$\begin{bmatrix}A&0\\C&D\end{bmatrix}=\begin{bmatrix}A&0\\0&I\end{bmatrix}\begin{bmatrix}I&0\\C&D\end{bmatrix}$ ∴ $\det\begin{bmatrix}A&0\\C&D\end{bmatrix}=\det A\cdot\det D$

$\det\begin{bmatrix}A&B\\0&D\end{bmatrix}=\det\begin{bmatrix}A&B\\0&D\end{bmatrix}^T=\det\begin{bmatrix}A^T&0\\B^T&D^T\end{bmatrix}=\det A^T\det D^T=\det A\det D$

应用. 用行列式推导 $\vec{u}\times\vec{w}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\u_1&u_2&u_3\\w_1&w_2&w_3\end{vmatrix}$

□ 考虑 $f\begin{bmatrix}x\\y\\z\end{bmatrix}=\det\begin{bmatrix}x&u_1&w_1\\y&u_2&w_2\\z&u_3&w_3\end{bmatrix}$ 可知它是线性的. 因 input $\mathbb{R}^3$ output $\mathbb{R}^1$, 故 $f(\vec{x})$ 可用 $A\vec{x}$ 表示 $A_{1\times3}$ (其余行的行列式都为0) $[A_1 A_2 A_3]\begin{bmatrix}x\\y\\z\end{bmatrix}$ 可看作是点乘, 几何意义为 $(A_1 A_2 A_3)$ 在 $(x\ y\ z)$ 上的投影乘 (x,y,z) 长度. 按第一列展开得 $\begin{cases}A_1x=(u_2w_3-u_3w_2)x\\A_2y=(u_3w_1-u_1w_3)y\\A_3z=(u_1w_2-u_2w_1)z\end{cases}$. 可知 (x,y,z) 换为 $(\vec{i},\vec{j},\vec{k})$ 时成为向量.
因此 $A\begin{bmatrix}\vec{i}\\\vec{j}\\\vec{k}\end{bmatrix}$ 为一个向量 定义为 $\vec{u}\times\vec{w}$ 就是右边这个行列式

例 求A $A^{adj}=\begin{bmatrix}3&5&4\\1&0&1\\2&1&1\end{bmatrix}$ find A^{-1}

□ $A^{-1}=\frac{\operatorname{adj}A}{\det A}=\frac{1}{6}\begin{bmatrix}-1&-1&5\\1&-5&1\\1&7&-5\end{bmatrix}$

$|A|=0$. 即线性 tranform 后面积变为0, 起压缩. 维度减少, 但这样就有一个 input 对多个 output, 不是一个方程可做的. 如压缩后b在这压缩后的面/线上, 有sol. 不在则 $\overline{Ax=b}$ 无解 (压缩)

Invertible 了 & eigen 1

P7 J3

1. $A_{n\times n}$, $\det A^T = \det A$. 即 col operation 与 row operation 效果相同

2. Cramer's Rule. $A_{n\times n}$, 可逆. For any $\vec b$ in $\mathbb{R}^n$, unique $\vec x$ is
$x_i = \frac{\det A_i(\vec b)}{\det A}$, $i=1,2..n$. where $A_i(\vec b)$ is A replacing i col by $\vec b$
$A\cdot I_i(\vec x) = A[\vec e_1 \cdots \vec x \cdots \vec e_n] = [A\vec e_1 \cdots A\vec x \cdots A\vec e_n] = [\vec a_1 \cdots \vec b \cdots \vec a_n] = A_i(\vec b)$
$\det A \cdot \det(I_i(\vec x)) = \det(A_i(\vec b))$, $\det(I_i(\vec x)) = \vec x$ 即得证. 只对唯一解适用, 无穷解求不出

3. A $\overset{\text{non-singular}}{\text{可逆}}$, $A^{-1} = \frac{\text{adj} A}{\det A}$
jth col of A^{-1} is $\vec x$ that satisfies $A\vec x = \vec e_j$, 而 A^{-1} 的 (i,j) entry 为 $x_i = \frac{\det A_i(\vec e_j)}{\det A}$
note $\det A_i(\vec e_j) = (-1)^{i+j}\det A_{ji} = C_{ji}$. 因此 A^{-1} 的整体为 $A^{-1} = \frac{1}{\det A}\underbrace{\begin{bmatrix} C_{11} & C_{21} & C_{n1} \\ C_{12} & C_{22} & C_{n2} \\ C_{1n} & C_{2n} & C_{nn} \end{bmatrix}}_{\text{adj} A}$
注意 $\text{adj} A \cdot A = A \cdot \text{adj} A = \det A \cdot I$ ($A^{-1}\det A = \text{adj} A$, 同乘A)

4. eigenvector of $A_{n\times n}$ is $\boxed{\text{non zero}}$ vector $\vec x$ s.t. $A\vec x = \lambda \vec x$ for some scalar λ 参见 [25]
λ eigenvalue of $A_{n\times n}$ if $\exists$ nontrivial sol $\vec x$ for $A\vec x = \lambda\vec x$.
可知, λ is eigenvalue iff $(A-\lambda I)\vec x = \vec 0$ has nontrivial sol. The set of all sols is Null$(A-\lambda I)$
This set a subspace of $\mathbb{R}^n$, eigenspace of A corresponding to λ ⟵

定理 1. eigenvalue of triangular matrix are entries on main diagonal 即 $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ λ只有1, 只有一个
□ 因 triangular det 为 diagonal 相乘

定理 2. 0 is eigenvalue iff A not invertible. [5] J_1
□ $A\vec v = 0v = 0$ ∴ $Av=0$ nontrivial sol $(v\neq 0)$ ∴ A 非 1-1, nullity $\neq 0$ 则非 full-rank

定理 3. DEF polynomail $f(t) \in P[F]$ split over F iff
$\exists\, a_1 .. a_n, c \in F$, $f(t) = c(t-a_1)..(t+a_n)$
(不一定 distinct)
eg. $\boxed{(t^2+1)}(t-2)$ 非 split over $\mathbb{R}$, 但 split over $\mathbb{C}$ $(t-i)(t+i)$
diagonalizable T characteristic polynomial split over F 也即: 若有非 distinct 特征值, 有 repeated. algebra multiplicity > 1
□ [25]. J_{1a} show $f(\lambda)$ 为 poly, 且讨论和 ind. of basis. ∴ 对某β, $[T]_\beta = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$, 对其 $f(\lambda) = (\lambda_1 - t)..(\lambda_n - t) = (-1)^n(t-\lambda_1)..(t-\lambda_n)$

* 即使 split 不一定 diagonalizable.

定理 4 (Cayley-Hamilton) $\forall$ square A satisfy characteristic func.
即若 $f(\lambda) = (-1)^n\lambda^n + k_{n-1}\lambda^{n-1} + .. + k_1\lambda + k_0$, 有 $(-1)^nA^n + k_{n-1}A^{n-1} + .. + k_1A + k_0I = 0$

Date.

例1 $A=\begin{bmatrix}4&-1&6\\2&1&0\\2&-1&8\end{bmatrix}$ eigenvalue = 2, find basis for eigenspace for $\lambda=2$

解 $A-2I=\begin{pmatrix}2&-1&6\\2&-1&6\\2&-1&6\end{pmatrix}$，∴ 对 $(A-2I)x=0$ $\left[\begin{array}{ccc|c}2&-1&6&0\\&&&0\\&&&0\end{array}\right]$ 有 $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_2\underbrace{\begin{bmatrix}1/2\\1\\0\end{bmatrix}}_{u}+x_3\underbrace{\begin{bmatrix}-3\\0\\1\end{bmatrix}}_{v}$ → basis

例2 $A_{n\times n}$ 的 每一行 sum = s，则 特征值 为 s；$B_{n\times n}$，如 每一列 sum = s，特征值也为 s

解 1° $A\begin{bmatrix}1\\\vdots\\1\end{bmatrix}=\begin{bmatrix}a_{11}+\dots+a_{1n}\\\vdots\\a_{n1}+\dots+a_{nn}\end{bmatrix}=s\begin{bmatrix}1\\\vdots\\1\end{bmatrix}$ 2° A^T 与 A 同 eigval

应用. dynamic system. $A=\begin{bmatrix}.95&.03\\.05&.97\end{bmatrix}$, dynamic system defined by $\vec{x}_{k+1}=A\vec{x}_k$, $\vec{x}_0=\begin{bmatrix}.6\\.4\end{bmatrix}$

解 $\det\begin{bmatrix}.95-\lambda&.03\\.05&.97-\lambda\end{bmatrix}\Rightarrow\lambda=1/0.92$ 此时 $\vec{v}_1\begin{bmatrix}3\\5\end{bmatrix}$ $\vec{v}_2\begin{bmatrix}1\\-1\end{bmatrix}$ 为特征向量，

$\vec{x}_0=c_1\vec{v}_1+c_2\vec{v}_2$ ($\vec{v}_1,\vec{v}_2$ basis for $\mathbb{R}^2$) $=[\vec{v}_1\ \vec{v}_2]\begin{bmatrix}c_1\\c_2\end{bmatrix}$

求得 $[c_1\ c_2]=[\vec{v}_1\ \vec{v}_2]^{-1}\vec{x}_0=\begin{bmatrix}3&1\\5&-1\end{bmatrix}^{-1}\begin{bmatrix}.6\\.4\end{bmatrix}=\begin{bmatrix}.125\\.225\end{bmatrix}$

$\vec{x}_1=A\vec{x}_0=c_1A\vec{v}_1+c_2A\vec{v}_2=c_1\vec{v}_1+c_2(.92)\vec{v}_2$；$\vec{x}_2=A\vec{x}_1=c_1\vec{v}_1+c_2(.92)^2\vec{v}_2$

∴ $\vec{x}_k=c_1\vec{v}_1+c_2(.92)^k\vec{v}_2=(.125)\begin{bmatrix}3\\5\end{bmatrix}+(.225)(.92)^k\begin{bmatrix}1\\-1\end{bmatrix}$，当 $k\to\infty$，$\vec{x}_k=.125\vec{v}_1$

A 可为 Markov chain 的 migration matrix，$\vec{x}_0$ 初始人口，$\vec{x}_k$ 为 k 年后人口，可知会稳定下来

例3 $A=\begin{bmatrix}1&3&3\\-3&-5&-3\\3&3&1\end{bmatrix}$ 求 A^k；$B=\begin{bmatrix}2&4&3\\-4&-6&-3\\3&3&1\end{bmatrix}$；$C=\begin{bmatrix}5&-8&1\\0&0&7\\0&0&-2\end{bmatrix}$ 图4 $P=\begin{pmatrix}1&-1&-1\\-1&1&0\\1&0&1\end{pmatrix}$

解 1° $\det(A-\lambda I)=0\begin{cases}\lambda_1=1 & v_1=[1\ -1\ 1]\\ 2 & v_2=[-1\ 1\ 0]\\ 2 & v_3=[-1\ 0\ 1]^T\end{cases}$ $\{v_{1,2,3}\}$ L.Ind. ∴ $A=PDP^{-1}$ ∴ $A^k=PD^kP^{-1}$, $D^k=\begin{pmatrix}1&&\\&(-2)^k&\\&&(-2)^k\end{pmatrix}$

2° B 有 $\lambda_1=1$, $v_1=\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $\lambda_2=\lambda_3=-2$, $v_2=\begin{pmatrix}-1\\1\\0\end{pmatrix}$ 非 $\mathbb{R}^3$ basis，不可 diagonalizable.

3° C 有 $\lambda=5,0,-2$ distinct，可对角化

例4 若 $AP=PD$, D diagonal, then nonzero col of P must be eigenvector of A. ✓ 见书的证明

A has n L.Ind eigenvectors，则 A^T 也是 ✓ 即 A 可对角化，$A^T=(PDP^{-1})^T=(P^T)^{-1}D^T(P)^T$，可对角化，有 n 个 L.Ind 特征向

可逆与可对角化无本质联系. $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ 可对角化但不可逆；$A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ 可逆但不可对角化

① A^{-1} 的特征值 为 λ^{-1} P

□ 令 $A^{-1}x = tx$, $t \neq 0$ (A 可逆)

则 由 $Ax = \lambda x$, $A^{-1}A^{-1}Ax = A^{-1}A^{-1}\lambda x$

$tx = A^{-1}x = \lambda A^{-1}tx = \lambda t^2 x$, $t = \lambda t^2$, $t = \frac{1}{\lambda}$

② 若 $A^2 = 0$, then 唯一的特征值 $= 0$ eigen 2

□ 令 $Ax = \lambda x$, $AAx = 0 = \lambda^2 x$, $\therefore \lambda = 0$

③ A^T 与 A 特征值相同 □ $f(\lambda) = \det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I)$

④. T^m 特征值 λ^m □ $T^m(v) = \lambda \cdots \lambda v$ ⑤ $I_n - A$ eigval 为 $1-\lambda_1 \ldots 1-\lambda_n$. ⑥ AB, BA eval 相同 ……

若AB eval $= \lambda$ $\lambda Bv = B \cdot (ABv) = (BA)(Bv)$ $\therefore$ BA eval $= \lambda$

2. λ is eigenvalue iff $\det(A - \lambda I) = 0$ (需有 free variable, 即非 full-rk, i.e. $\det = 0$) (characteristic polynomial) 证明见图.2

(如为 $(\lambda-2)^2 = 0$ $\lambda_1 = \lambda_2 = 2$, 称为 multiplicity $= 2$)

3. $A, B_{n \times n}$, A is similar to B iff $\exists$ 可逆 P, s.t $P^{-1}AP = B$ or $A = PBP^{-1}$

changing $A \to P^{-1}AP$ is similarity transformation 即 change of basis 的 2 个 matrix 互相…… 见 T_{2a}

If A, B similar, they have same eigenvalue 同 algebraic/geometric multiplicity, 但 evec diff ＊同 eigval 不一定 similar,

＊ share trace: $tr(B) = tr(P^{-1}AP) = tr(PP^{-1}A) = trA$

□ 因 $B = P^{-1}AP$ $\det(B - \lambda I) = \det(P^{-1}AP - \lambda P^{-1}IP) = \det(P^{-1}(A - \lambda I)P) = \det P^{-1} \det(A - \lambda I) \det P = \det(A - \lambda I)$

4. 令 $E_\lambda = \{x \in V \mid T(x) = \lambda x\}$ eigenspace of λ, 可知 $E_\lambda \leq V$

即 $Nul(T - \lambda I)$!! 为 eigenvec 和 zero 非 eigenvec

T_1 对 $T: V \to V$, λ algebra multiplicity m 即 $f(t)$ characteristic poly 中的 $(t-\lambda)^m$, 有 $1 \leq$ geometric multiplicity $= \dim(E_\lambda) \leq m$

□ 令 E_λ basis $\{v_1 \cdots v_p\}$. 因 $E_\lambda \leq V$, extend 为 $\{v_1 \cdots v_p, \cdots v_n\}$ basis for V (＊＊＊3. T_3). 考虑 $[T]_\beta = A$. 应用

note 对前 p: $[\underset{= \lambda v_i}{T(v_i)}]_\beta = \begin{bmatrix} 0 \\ \lambda \\ 0 \end{bmatrix}$. $\therefore A = \begin{bmatrix} \lambda I_p & B \\ 0 & C \end{bmatrix}$. 而 $f(t) = \det\begin{pmatrix} (\lambda - t)I_p & B \\ O & C - tI_{n-p} \end{pmatrix} = \det(\lambda - t)I_n \cdot \det(C - tI_{n-p})$

$= (\lambda - t)^p \cdot \underline{\det}$ 与t无关的poly, 因 $(\lambda - t)^p$ 为 $f(t)$ 一个 factor, $\therefore m \geq p = \dim(E_\lambda)$

T_2 令 $\lambda_1 \cdots \lambda_k$ distinct 特征值 for T, 令 S_i E_{λ_i} 的 L.ind. subset. $S = \bigcup_{i=1}^{k} S_i$ L.ind. subset of V

□ 令 $S_i = \{v_{i1} \cdots v_{in_i}\}$, 考虑 $a_{11}v_{11} + \cdots + a_{1n_1}v_{1n_1} + \cdots + a_{k1}v_{k1} + \cdots + a_{kn_k}v_{kn_k} = 0 = \sum_{i=1}^{k}\sum_{j=1}^{n_i} a_{ij} v_{ij}$

但对各 i, 令 $w_i = \sum_{j=1}^{n_i} a_{ij}v_{ij} \in E_{\lambda_i}$, 这样 $w_1 + \cdots + w_k = 0$, 表明 $w_i = 0$ (若不, 有 $w_{n_1} \ldots w_{n_m}$ 非 0, 则对这 m 个 eig.vec. 因 T_{2a}, L.ind. $\therefore w_{n_i} = 0$ 矛盾). 而 S_i L.ind. $\therefore a_{ij} = 0$ for all j

对 $A_{n \times n}$, 如 distinct eigenvalue 为 $\lambda_1 \ldots \lambda_p$: ($p < n$)

a. $1 \leq k \leq p$, $1 \leq \dim$ Eigenspace λ_k $\leq$ multiplicity of λ_k

b. A 可对角化 iff sum of dim of eigenspace $= n$, 这只有 i) 特征多项式 ~~[illegible]~~ 可被 split over ii) 每个 λ_k 的 dim ES $= \lambda_k$ 的 multiplicity 这二者同时满足才可能

c. 如 A 可对角化 且 β_k 是 dim ES λ_k 的 basis, 则 $\beta_1 \ldots \beta_p$ 构成所有向量 forms an eigenvector basis for $\mathbb{R}^n$

(split over: $t^2 - 1$ ✓ 因 $= (t+1)(t-1)$, 但 $t^2 + 1$ 不可, 可对角化 A 的特征多项式 splits, 但反之不对, 除非 b.ii))

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vs 6

Date. /

1. Subspace 可视为 set of all sols to a system of homogeneous linear equation → Null space ①
也可视为 set of all L. Comb. of certain specified vectors → Col space ②

①: $A\vec{x}=\vec{0}$, $\vec{x}=0$ 肯定在; $\vec{u},\vec{v}$ in Null, $A(\vec{u}+\vec{v})=A\vec{u}+A\vec{v}=\vec{0}$; $cA\vec{u}=Ac\vec{u}=\vec{0}$

②: 见前几页, 令 $\vec{v}_1,\dots,\vec{v}_p$ span Col, weight 全0 则为 $\vec{0}$; $\vec{u},\vec{v}$ 相加也可视为 L. Comb; $c\vec{u}$ 只是乘以 weight
↳ Col space 被 span 出

从求 Nul A 的过程中, 得知 ① 可转化为 ② (将 ① 中 sol 最后变为 sol 是一些向量 L. Comb)

①② 从代性变换来看, ① 为 Kernal, ② 为 range

例 $\left\{\begin{bmatrix}a\\b\\c\end{bmatrix} : a+b+c=2\right\}$ 不为 subspace 非 homogeneous equation 无0　　$\left\{\begin{bmatrix}b-2d\\5+d\\b+3d\\d\end{bmatrix} : b,d\in\mathbb{R}\right\}$ ✗ 有5在常数, 不能写成 L. Comb 无0

$\left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix} : \begin{matrix}a-2b=4c\\2a=c+3d\end{matrix}\right\}$ ✓ 记为 Nul A, $A=\begin{bmatrix}1&-2&-4&0\\2&0&-1&-3\end{bmatrix}$　　$\left\{\begin{bmatrix}c-6d\\d\\c\end{bmatrix} : c,d\in\mathbb{R}\right\}$ ✓ $\begin{bmatrix}1\\0\\1\end{bmatrix}c+\begin{bmatrix}-6\\1\\0\end{bmatrix}d$ 为 Col A, $A=\begin{bmatrix}1&-6\\0&1\\1&0\end{bmatrix}$

例 用 subspace 解释 为何 $\begin{cases}x_1-3x_2-3x_3=0\\-2x_1+4x_2+2x_3=0\\-x_1+5x_2+7x_3=0\end{cases}$ 已知解为 $x_1=3, x_2=2, x_3=-1$ 时, 解的倍数也为解

□ 可记为 Nul A, $A=\begin{bmatrix}1&-3&-3\\-2&4&2\\-1&5&7\end{bmatrix}$, $\vec{x}=\begin{bmatrix}3\\2\\-1\end{bmatrix}$ in Nul A, $c\vec{x}$ 也在 (Nul A 为 subspace)

同理, 用 Col Space 可知 $\begin{cases}5x_1+x_2-3x_3=0\\-9x_1+2x_2+5x_3=1\\4x_1+x_2-6x_3=9\end{cases}$ 有解时, $\begin{cases}5x_1+x_2-3x_3=0\\-9x_1+2x_2+5x_3=5\\4x_1+x_2-6x_3=45\end{cases}$ 也有解. ($5\begin{bmatrix}0\\1\\9\end{bmatrix}=\begin{bmatrix}0\\5\\45\end{bmatrix}$, 也在 Col A)

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lagrange

应用 $\mathbb{P}_n$ 为 polynomial space (见P63), $\vec{p}_n(t)=a_0+a_1t+\dots+a_nt^n$ 可证为 Vector Space, 在 statistical trend analysis 用到

(lagrange Interpolation) 对 $c_0\dots c_n\in\mathbb{F}$ 且 distinct, DEF $f_i(x)=\frac{(x-c_0)\dots(x-c_{i-1})(x-c_{i+1})\dots(x-c_n)}{(c_i-c_0)\dots(c_i-c_{i-1})(c_i-c_{i+1})\dots(c_i-c_n)}$
$=\prod_{k=0,\,k\neq i}^{n}\frac{x-c_k}{c_i-c_k}$, 证 $\{f_0\dots f_n\}$ 为 $\mathbb{P}_n(\mathbb{F})$ 的 basis

□ $\sum_{i=0}^{n}a_if_i$ = zero poly, 即 $\forall x$ poly$=0$, $\therefore \sum_{i=0}^{n}a_if_i(c_j)=0$ 对 $j=0\dots n$
$\sum a_if_i(c_j)=a_j$ (只有当 $i=j$ 才为1)
但 $f_i(c_j)=\begin{cases}0 & i\neq j\\1 & i=j\end{cases}$, ∴ 对各 $a_j=0$ 而 $\dim(\mathbb{P}_n)=n+1$, ∴ $\{f_0\dots f_n\}$ 这 n+1 L.Ind 为 basis　5.Pd

∴ 因此对 poly $p(x)\in\mathbb{P}_n(\mathbb{F})$, $p(x)=\sum_{}^{n}a_if_i$, 而对 $j=0\dots n$, $p(c_j)=\sum a_if_i(c_j)=a_j$
$\therefore p=\sum_{j=0}^{n}p(c_j)f_j$

可知 对 poly 若其对已知的 n+1 个 distinct c_i, $p(c_i)$ 都$=0$.
则 $p=\sum 0f_i=0$, zero func!!

eg1 已知 poly 经过 (1 8) (2 5) (3 -4), 找出 poly 的拟合.
□ 对 poly p $\begin{cases}p(1)=8\\p(2)=5\\p(3)=-4\end{cases}$ 选 $\begin{matrix}c_0=1\\c_1=2\\c_2=3\end{matrix}$ 则 $f_0=\frac{(x-2)(x-3)}{(1-2)(1-3)}$　$p=8\cdot f_0+5f_1+4f_2=-3x^2+6x+5$

eg2. $c_0\dots c_n$ distinct, def $T:\mathbb{P}_n(\mathbb{F})\to\mathbb{F}^{n+1}$, $f\to\begin{pmatrix}f(c_0)\\\vdots\\f(c_n)\end{pmatrix}$, 证 T isomorphism
□ linearity 易证, 且 dim 相同 只需 show 1-1 (四.4) 令 $T(f)=0$, ∴ 对各 c_i, $f(c_i)=0$ ∴ f 为 zero vec.

应用 Verify $S=\{1,t,t^2,\dots,t^n\}$ is a basis for $\mathbb{P}_n$. (S 为 standard basis)

□ S span $\mathbb{P}_n$; $c_0\cdot1+c_1\cdot t+\dots+c_n\cdot t^n=\vec{0}(t)=0$. 代数上 多于 n个0点的多项式 …… 式
因此 $c_0=0=c_1=c_2\dots$, S L.Ind

***2

Span 3

set 内∀ vector 可由 basis 唯一表示

1-(The spanning set theurum) 若 finite set $S=\{\vec{v}_1,\dots\vec{v}_p\}$, span(S) = Vec. Space V, 则 S 的某 subset 为 V 的 basis

□ 若 $S=\phi$ or $\{0\}$, span(S) 和 $V=\{0\}$, S 均为 basis. 若 S 有非 0 项, 构造 basis by 步骤 ⓐ:

ⓐ: pick 非0 $\vec{v}_1$, note $\{\vec{v}_1\}$ L. Ind, 若可能, 找剩余的加上可构成 L. Ind. 的 $\vec{v}_i$ … 直到找到 $\beta=\{u_1,\dots,u_k\}$ L. Ind.

WTS β 为 basis, 即 β spans V: i) $span(\beta)\subseteq V$ 因 $\beta\subseteq S$, $\underline{span(\beta)\subseteq span(S)}^{P_2 P\,i)}=V$

ii) $V\subseteq span(\beta)$, 若 $v\in V$ 也在 β 内, 自然 $v=1\cdot v\in span(\beta)$; $v\in V-\beta$, 若 $v\notin S$, v 为 S 的 L.comb 表示, 同 $v\in S$,

则: $\beta\cup\{v\}$ 为 L. dep (否则 @会继续), 由 ☐.4 #2 推得 $v\in span(\beta)$

eg: 有 $S=\{\begin{pmatrix}2\\-3\\5\end{pmatrix}\begin{pmatrix}8\\-12\\20\end{pmatrix}\begin{pmatrix}1\\0\\-2\end{pmatrix}\begin{pmatrix}0\\2\\-1\end{pmatrix}\begin{pmatrix}7\\2\\0\end{pmatrix}\}$ span $\mathbb{R}^3$. pick $\begin{pmatrix}2\\-3\\5\end{pmatrix}$, $\begin{pmatrix}8\\-12\\20\end{pmatrix}=2\begin{pmatrix}2\\-3\\5\end{pmatrix}$ 不 include, $\begin{pmatrix}1\\0\\-2\end{pmatrix}$ 可 include, $\begin{pmatrix}0\\2\\-1\end{pmatrix}$ 加上 $\{\begin{pmatrix}2\\-3\\5\end{pmatrix}\begin{pmatrix}1\\0\\-2\end{pmatrix}\}$

or: 即找出 free variable, $\begin{bmatrix}2&8&1&0&7\\-3&-12&0&2&2\\5&20&-2&-1&0\end{bmatrix}\to\begin{bmatrix}1&4&&&2\\&&1&&3\\&&&1&4\end{bmatrix}$ 有 1st, 3rd, 4rd 101 $\begin{pmatrix}7\\2\\0\end{pmatrix}$ exclude, ∴ basis = $\{\begin{pmatrix}2\\-3\\5\end{pmatrix},\begin{pmatrix}1\\0\\-2\end{pmatrix}\begin{pmatrix}0\\2\\-1\end{pmatrix}\}$

例 find basis for $x+2y+z=0$ in $\mathbb{R}^3$.

□ 可记为 sol for homogeneous equation. Null A, A = [1 2 1], $\{[\,\vdots\,],[\,\vdots\,]\}$

let $\vec{v}_1=\begin{bmatrix}1\\0\\1\end{bmatrix}$ $\vec{v}_2=\begin{bmatrix}0\\1\\1\end{bmatrix}$ $\vec{v}_3=\begin{bmatrix}0\\1\\0\end{bmatrix}$, let H be set of vectors in $\mathbb{R}^3$ whose 2nd 3rd entry are equal

Then $\forall\vec{x}$ in H is L.Comb of $\vec{v}_1\vec{v}_2\vec{v}_3$, $\begin{bmatrix}s\\t\\t\end{bmatrix}=s\begin{bmatrix}1\\0\\1\end{bmatrix}+(t-s)\begin{bmatrix}0\\1\\1\end{bmatrix}+s\begin{bmatrix}0\\1\\0\end{bmatrix}$

但 $\{\vec{v}_1\,\vec{v}_2\,\vec{v}_3\}$ 不是 basis for H, 因 $\vec{v}_1\,\vec{v}_2\,\vec{v}_3$ 甚至不在 H 里, 基是在 H 里的向量, 且 L. Ind 且 span

比较 ~~10~~ $\overline{T}_\beta$, 对一个 given T, $[T]^r_\beta$ 总 unique, 见基于, 而 ~~10~~ 10 的 Thm show 对 given output, 有且只有一个 T st. basis → output

2、Coordinate (线性关系也保证) ☐.2 例题见 ☐ Ex5

$\beta=\{\vec{b}_1,\dots\vec{b}_n\}$ 为 V 的 basis, 对 $x\in V$, 若 $x=\sum^n a_i\vec{b}_i$, 令 coord $[x]_\beta$ 为 $\begin{bmatrix}a_1\\\vdots\\a_n\end{bmatrix}$, 事实上,

对. $T: V\to\mathbb{F}^n$, $x\to[x]_\beta$ 不仅 linear 且 bijective

□ 对 $x=\sum^n\alpha_i b_i$, $y=\sum^n\beta_i b_i$ $T(x+ky)=\begin{bmatrix}\alpha_1+k\beta_1\\\vdots\\\alpha_n+k\beta_n\end{bmatrix}=T(x)+kT(y)$, ∴ linear; 1-1, 令 $T(x)=0=\begin{bmatrix}0\\\vdots\\0\end{bmatrix}$ ∴ $a_i=0$ ∴ $x=\sum^n 0b_i=0$ ∴ Nullity = 0 1-1

onto: note dim V = n = dim F, 由 ☐ dim 相同的 1-1 即 ⇒ onto

推广, 已用 col vector 表示 Vector space 的一 ele, 用 matrix 表示 其 L. T. (即 basis 被映射的 $\overset{\frown}{x}$ 即可表示) ($T(v_j)\in W$, 可被 r 表示) (a_{ij} i: 1→m unique, 由 ~~...~~ 3. T_1)

对 $T: V\to W$, $\beta=\{v_1,\dots,v_n\}$, $r=\{w_1,\dots,w_m\}$, 对 v_j (j: 1→n) $\to\sum_{i=1}^m a_{ij}w_i$

记 a_{ij} 组成的 matrix (m×n) 为 $[T]^r_\beta$, 对 $[x]_\beta$, $T([x]_\beta)$ 为 $[x]_r$, $[T]^r_\beta=[[T(v_1)]_r,\dots,[T(v_n)]_r]$ 分析 a b c ⟩

对 $x=\sum x_i v_i$, $T(x)=\sum x_i T(v_i)$, 同 $[T]^r_\beta[x]_\beta=[[T(v_1)]_r\dots]\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}=x_1[T(v_1)]_r+\dots$

证明 L. Ind 时可用 $af_1(x)+bf_2(x)+cf_3(x)=0$ 取 3 个 x 值

总 ~~3 证明见 ☐ 4~~ ∴ 总可用 matrix 研究 ∀ vector space, 这样证明 to $\mathbb{R}^n$ book

图 这种 coordinate mapping 是一种 isomorphism from V to $\mathbb{R}^n$. 虽然可能维度不同, 但 "act alike"

例 i) $T:\mathbb{R}^3\to\mathbb{R}^2$, $(a_1,a_2,a_3)\to(2a_1+3a_2-a_3, a_1+a_3)$, 找 $[T]^r_\beta$, ii) $T:\mathbb{R}^2\to\mathbb{R}^3$, $\begin{bmatrix}a_1\\a_2\end{bmatrix}\to\begin{bmatrix}a_1-a_2\\a_1\\2a_1+a_2\end{bmatrix}$, $\beta=\{\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}\}$, $r=\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}2\\2\\3\end{bmatrix}\}$

□ i) 有 $[\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}3\\0\end{bmatrix}_r,\begin{bmatrix}-1\\1\end{bmatrix}]=\begin{bmatrix}2&3&-1\\1&0&1\end{bmatrix}$ ii) $[\begin{bmatrix}-1\\1\\4\end{bmatrix}_r\begin{bmatrix}-1\\2\\7\end{bmatrix}_r]=\begin{bmatrix}-7/3&-11/3\\2&3\\2/3&4/3\end{bmatrix}$

3-DEF $\mathcal{L}(V,W)$ 为 all L.T. from V to W, 可证 其为 V.S. (addition, scalar multiply 为 of f+g 六种证明法)

T (矩阵表示法也 linear) 对 $T,U\in\mathcal{L}(V,W)$ 有 $[T+U]^r_\beta=[T]^r_\beta+[U]^r_\beta$, $[kT]^r_\beta=k[T]^r_\beta$

□ i) 有 a_{ij}, b_{ij} st. $T(v_j)=\sum_i a_{ij}w_i$, $U(v_j)=\sum_i b_{ij}w_i$, $(T+U)(v_j)=\sum(a_{ij}+b_{ij})w_i$ ∴ 左 $=a_{ij}+b_{ij}=$ 右

ii) $kT(v_j)=k\sum a_{ij}w_i$ ∴ 左 $=ka_{ij}=$ 右

***3

VS 7 & dim 3

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BEF见回1

T_1 $\beta=\{s_1,\dots,s_n\}$ 为 V.S. 的 subset. β为 basis ⇔ $\forall v\in V$, v为β的 unique L.comb..

□ ⇒ 见 P2 P_{iv} ⇐ i) β L.Ind. 对 $\vec{0}\in V$, 唯一表示 即 trivial 表示, ∴ coeff = 0
ii) span, trivial since ∀v 有 L.comb

T_2 (Replacement) 对 V.S. V, 有 n-ele G, span(G)=V, 有 m-ele L.Ind. L 且 $L\subseteq V$. 证 $\exists H\subseteq G$, n-m ele (m≤n 且), 且 $L\cup H$ span V
回 #6

□ induction on m: m=0, L=φ, take H=G; 对 m+1, $L=\{v_1,\dots,v_{m+1}\}$, 因 $\{v_1,\dots,v_m\}$ L.Ind 由 inductive hypothesis, n-m ele
有 $\{u_1,\dots,u_{n-m}\}\subseteq G$ 且 $\{v_1,\dots,v_m\}\cup\{u_1,\dots,u_{n-m}\}$ ② span V, 即: 对 $v_{m+1}\in V$, $a_1v_1+\dots+a_mv_m+b_1u_1+\dots+b_{n-m}u_{n-m}=v_{m+1}$ ①
$n-m>0$ (若 $n-m=0$, $v_{m+1}\in span(\{v_1,\dots,v_m\})$, $L=\{v_{m+1}\}\cup\{v_1,\dots,v_m\}$ L.dep, 矛盾 回.4.#2), ∴ 同理 b_i 不全 0, 令 $b_1\neq 0$, $u_1=(-b_1^{-1}a_1)v_1+\dots$
即 n-m 个, n-(m+1) 对 n-m-1 ele $H=\{u_2,\dots,u_{n-m}\}$, WTS $span(L\cup H)=V$. i) ⊆: $L\cup H\subseteq V$, span ⊆ V (P2 T_1) ii) ⊇: 由① $u_1\in span$, ② ⊆ span, ∴ span(②) = V ⊆ LHS

P Replacement 的推论

a.) If vector space V basis $\beta=\{b_1,\dots,b_n\}$, then any set in V with more than n vectors is L.dep
回 #6

□ 法1. 若 L.Ind, 对大于n的 set 有 subset S, 有 n+1 ele. 且 S 也 L.Ind. 由 T_2, n+1 ≤ n, 矛盾

法2. col > row, 有 free variable, L.dep 回.4.T#3

b) V 的 basis n vector, then ∀ basis has n vectors

□ 若 β_1 β_2 为 basis, β_1 n vector, β_2 m vectors, 若 m > n, 由 a) β_2 L.dep. ∴ m ≤ n. 对 β_1 若 n > m, 同理 n ≤ m ∴ m = n

c.) ∀ L.Ind. subset L 可被 expand 为 V 的 basis

□ 若 dim(V)=n, |L|>n. 由 a) L dep; 若 |L|≤n, 有 $H\subseteq\beta$ (T_2) st $L\cup H$ span V, $|L\cup H|$ at most n

d.) (basis theorem) dim(V)=n, any V 的 spanning set at least n ele, 若 n ele 则为 V basis;

any L.Ind L 有 n ele 为 V 的 basis b)

又 n个 vector in $\mathbb{R}^n$ span $\mathbb{R}^n$ 也为 basis 若不是 有 v_k L.dep, ∴ $span\{v_i\}=span\{v_i\}\backslash v_k$ 但 n-1 vec 不可能 span $\mathbb{R}^n$

□ i) let G spanning set. $H\subseteq G$ 为 basis (***2.1), |H|=n. 由 |G|≥|H| (subset), |G|≥n. 若 |G|=n, G=H

ii) 由 c) 扩充为 spanning set, 有 n-n=0 subset H. st. $L\cup H$ = L span V. ∴ L span 且 L.Ind. L 为 basis

T_3 $W\leq V\Rightarrow dim(W)\leq dim(V)$, 且 basis for W 可 extend 为 basis for V 已知: 等号当 W=V 成立

□ i) let dim(V)=n. if W={0}, dim(W)=0≤n; 若 if $\exists x_1\in W$ 且 $x_1\neq 0$, 由 ***2.1 步骤④ 构造
L.Ind. $\{x_1,\dots,x_k\}$, 由 $P_{a)}$ k≤n 不到 L.Dep. 余下的 $x_i\cup\{x_1,\dots,x_k\}$ L.dep, ∴ $x_i\in span(x_1,\dots,x_k)$, ∴ $span(x_1\dots x_k)=W$, basis
ii) 由 $P_{b)}$

例1 对 $\{u_1\cdots u_s\}$, $\{v_1\cdots v_t\}$ s>t, u_i 为 V 的 L.comb. 证 U L.dep 找T_3 找$P_{d)}$

□ span U ⊆ span V, 且 span U, span V 为 vec. space (P2 T_1) ∴ span U ≤ span V, $dim(\langle U\rangle)\leq dim(\langle V\rangle)\leq t<s$
回.T#3 的结论, 但因 V 不是 L.Ind, 不可写为 $[[u_1]_V\dots]$ 然后说 col # > row # $dim(\langle U\rangle)<S$, 必有ele 可被 L.comb. ∴ L.dep

例2 $\{\vec{v}_1\cdots\vec{v}_p\}$ span V, dim V ≤ p ✓ ; $\{\vec{v}_1\cdots\vec{v}_p\}$ L.Ind in V, dim V ≥ p ✓

$\{\vec{v}_1\cdots\vec{v}_p\}$ L.dep in V, dim V ≤ p ✗ ; If any set of p element in V fail to span V, dim V > p ✓

与 L.d 无关, 如全是 0

例3 $\alpha_1\cdots\alpha_3$ L.dep, $\alpha_2\cdots\alpha_4$ L.Ind; 证 $\alpha_1\in\langle\alpha_2\,\alpha_3\rangle$, $\alpha_4\notin\langle\alpha_1\,\alpha_2\,\alpha_3\rangle$

□ 有不全为0 c_i st. $c_1\alpha_1+\dots+c_3\alpha_3=0$, $c_1\neq 0$, 否则 $\alpha_2\,\alpha_3$ L.dep ∴ $\alpha_1=\frac{-c_2}{c_1}\alpha_2+\frac{-c_3}{c_1}\alpha_3$
若 $\alpha_4\in span(\alpha_1\cdots\alpha_3)=span(\alpha_2\,\alpha_3)$ (P2 $P_{c)}$), 不 L.Ind.   方法类似 回T#2

*** 4

Rank & basis

Date. /

Rank 1

T_3 A_{nm} rk = 1 ⟺ ∃ $u \in \mathbb{R}^n$ $v \in \mathbb{R}^m$, $uv^T = A$

□ ⇐ 对 $x \in \mathbb{R}^m$, $Ax = uv^Tx = (v^Tx)u$, 即 range map ∀x to span{u}

⇒ ∴ ∀x ∈ $\mathbb{R}^m$ 有 $Ax = k\vec{u}$, for some $\vec{u} \in \mathbb{R}^m$ ∴ 对 x = natural basis e_i 成立, ∴ $A = (k_1 u, k_2 u \ldots k_m u)$ $= u(k_1 \ldots k_m) = uv^T$

应用: 40 equation 42 variable to homogeneous system 有2解(不是multiply), 其它均可被这2解表达, 证 对 nonhomogeneous system with same coefficient 有解.

□ $A_{40\times42}$ 因 Ax = 0 有2 Lind sol, nullity = 2, rk A = 40. ∴ Col A = $\mathbb{R}^{40}$. ∀b, Ax = b 有 s-l

T_4 A rk = k, 则 $A = \sum^k$ rk-1-matrix □ $A \overset{SVD}{=} U\Sigma V^T$, 令 $\Sigma = \sum^k \boxed{\Sigma_i}$ ∴ $A = \sum^k U\Sigma_i V^T$

Σ_i: diag(0..σ_i 0..0)

T_5 (Eckart-Young) 对 rk = r A_{mn} 令 rk = k B_{mn} 对 k < r, 定义 A 的 rk-k approxi. $\hat{A}(k) = \sum_{i=1}^k \sigma_i \vec{u}_i \vec{v}_i^T$, 满足

i) $\hat{A}(k) = \underset{rk B = k}{\operatorname{argmin}} \|A - B\|_2$ 即 rk A approxi 在 ℓ_2 matrix norm 下 最优

ii) $\|A - \hat{A}(k)\|_2 = \sigma_{k+1}$

应用: Rank 1 矩阵的算法/singular value decomposition 有用

Let $A_{2\times3}$, rank A = 1 $\vec{u}_1$ first col of A, $\vec{u}_1 \neq 0$. 证 必有 $\vec{v}$ in $\mathbb{R}^3$ s.t. $A = \vec{u}\vec{v}^T$

□ rank A = 1, $\vec{u}_1$ 为 basis for Col A, $\vec{u}_2 = x\vec{u}_1$, $\vec{u}_3 = y\vec{u}_2$, $\vec{v} = \begin{bmatrix}1\\x\\y\end{bmatrix}$

(如 $\vec{u}_1 = \vec{0}$, 而 $\vec{u}_2$ 为 basis, 令 $\vec{u}_3 = x\vec{u}_2$, $\vec{v} = \begin{bmatrix}0\\1\\x\end{bmatrix}$; $\vec{u}_1, \vec{u}_2 = \vec{0}$, $\vec{u}_3$ 为 basis, $\vec{v} = \begin{bmatrix}0\\0\\1\end{bmatrix}$)

PP $A_{m\times n}$ 有 rank 1 iff outer product 存在, i.e. $A = \vec{u}\vec{v}^T$ for $\vec{u}$ in $\mathbb{R}^m$, $\vec{v}$ in $\mathbb{R}^n$

$x = a\beta_1 + b\beta_2 + c\beta_3$
新的basis r_1 r_2 r_3 新的x 坐标为 $ar_1 + br_2 + cr_3$ $(r_1, r_2, r_3) = Q\begin{pmatrix}a\\b\\c\end{pmatrix}$

事实上 本质也只是 identity map ← 并不会对 value 做什么, 而是只是转换为另一basis 的表示

3. Basis change

若 given V, 有 $\beta = \{x_1 \ldots x_n\}$, $\beta' = \{x_1' \ldots x_n'\}$. 为了找出 互换 coord 的 matrix, def $T: x_i \to \sum_{i=1}^{n} Q_{ij} x_i'$

T_1 let β, β' two basis for V, let $\underset{\beta'\to\beta}{Q} = [I_V]_{\beta'}^{\beta}$, 则 Q invertible 且 ∀ v ∈ V. $[v]_\beta = \underset{\beta'\to\beta}{Q}[v]_{\beta'}$

□ note I_V bijection, thus ∃ inverse. 因.1 定理, Q invertible and $[v]_\beta = [I_V(v)]_\beta = [I_V]_{\beta'}^{\beta}[v]_{\beta'}$. 因 4

之所以 会想 def Q 为 I_V 的 matrix 原因, 因 想实现 对各v, $[v]_{\beta'} \to [v]_\beta$ 而 $[v]_\beta = [T(v)]_\beta = Q[v]_{\beta'}$ 只有 identity 才能做到. 因.4

若已有 T: V→V 在 β 中, 想知道 T 在另一basis 的表现形式 expression: 因.1 末

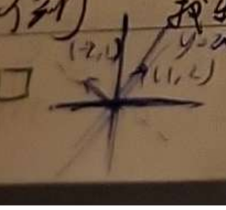
T_{2a} $[T]_{\beta'} = \underset{\beta'\to\beta}{Q^{-1}}[T]_\beta \underset{\beta'\to\beta}{Q}$ 两者 similar

所以右→左读, β'表示→β→β'

□ $Q[T]_{\beta'} = [I_V]_{\beta'}^{\beta}[T]_{\beta'} = [I_V T]_{\beta'}^{\beta} = [T \cdot I_V]_{\beta'}^{\beta} = [T]_\beta Q$

例) 找出 沿 y=2x 对称 翻转的 linear T 矩阵

□



找 (-2,1) (1,2) 来构造, note (1,2) (-2,1) linear Indpdnt 且 T(1,2)→(1,2) T(-2,1)→(2,-1) 全基为 β'

$[T]_{\beta'} = \underset{\beta'\to\beta}{Q^{-1}}[T]_\beta \underset{\beta'\to\beta}{Q}$

$\begin{bmatrix}1&0\\0&-1\end{bmatrix} = Q^{-1}?\begin{bmatrix}1&-2\\2&1\end{bmatrix}$

$? = \begin{bmatrix}-0.6 & 0.8\\0.8 & 0.6\end{bmatrix}$ ∴ $T(a\ b) = ?\begin{pmatrix}a\\b\end{pmatrix}$

***5 basis 2 & Decomposition

即找 $\sum c_i v_i' = v_1$ 或解 $[v_1' \cdots v_n'] \begin{bmatrix} c_1 \\ \vdots \end{bmatrix} = v_1$, $[v_1]_{\beta'} = [v_1' \cdots v_n']^{-1}[v_1 \cdots v_n]$ 例.1

当 β, β' 相同时, 使用 例9 2.3例1.

$\beta = \{v_1 \cdots v_n\}$ $\beta' = \{v_1' \cdots v_n'\}$

OR $\underset{\beta\to\beta'}{Q} = [[v_1]_{\beta'}, \dots] = [[v_1' \cdots v_n']^{-1} v_1, \dots] = [v_1' \cdots v_n']^{-1}[v_1 \cdots v_n]$ 见 T_3

$x = \sum x_i v_i = [v_1 \cdots v_n][x]_\beta$
$= \sum x_i' v_i' = [v_1' \cdots v_n'][x]_{\beta'}$

已知 $P = [I_W]_{r'}^{r}$, $P^{-1} = [I_W^{-1}]_{r}^{r'}$

Def T_{2d} $[x]_{\beta'} = [v_1' \cdots v_n']^{-1}[v_1 \cdots v_n][x]_\beta$ □

T_{2b} $T: V \to W$, 则 $[T]_{\beta'}^{r'} = \underset{r\to r'}{P^{-1}} [T]_{\beta}^{r} \underset{\beta'\to\beta}{Q}$ □ $[T]_{\beta'}^{r'} = [I_W T I_V]_{\beta'}^{r'} = [I_W]_{r}^{r'}[T]_{\beta}^{r}[I_V]_{\beta'}^{\beta} = P^{-1}[T]_{\beta}^{r} Q$

note 若 $T: V \to U$ 有 $[T]_{\beta}^{r} = \underset{\beta\to r}{P}[T]_{\beta}$

T_{2c} $\underset{\beta\to r}{R}\,\underset{\alpha\to\beta}{Q} = \underset{\alpha\to r}{S}$ □ $[I_V]_{\beta}^{r}[I_V]_{\alpha}^{\beta} = [I_V]_{\alpha}^{r} = \underset{\alpha\to r}{S}$

应用 LU Factorization (pd 为 Cholesky, Option P7.1)

1. If $A_{m\times n}$ 可在无 row interchange 时化为 echelon form, 则

$A_{mn} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}\begin{pmatrix} \square & & \\ 0 & \ddots & \\ & & \square \end{pmatrix}_A$

L_{mm} lower △ U_{mn} echelon form

可见当 U 若 diagonal = zero, A 非可逆

□ 即 A 可由 elementary row operation reduce 为 U, $E_p E_{p-1} \cdots E_1 A = U$, ∴ $A = (E_p \cdots E_1)^{-1} U$ more details 见 P1 b)

L, product of 上△ 为上△

P i) LU not unique, 但若 require $diag(L) = (1 .. 1)$, unique □ 若 $D \neq I_n$, $A = LU = LDD^{-1}U$ 新 U, 上△

新 L, lower △

若 LD lower △, D lower △; 可 $D^{-1}U$ 仍是上△ ∴ D 只能 diag matrix 由 $D \neq I_n$, 可 $D_{ii} \neq 1$, 但 $(LD)_{ii} = \sum_j L_{ij} D_{ji} = D_{ii} \neq 1$ diag(L)=(1..1)

可新 L 需 diag = (1..1), ∴ 不存在 D

ii) $\det A_{nn} = \prod_{i=1}^{n} U_{ii}$ □ $\det L \det U = 1 \cdot \prod U_{ii}$ 上/下△ 可 Π diag 见.1

iii) solve Sys Eq $Ax = b$ 方便: i) solve $Ly = b$ 用 forward/backward sub 因 L△, ii) solve $Ux = y$

iv) 非 ∀ matrix 有 LU $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. 若有 LU 可知 at least 1 个 L, U 非可逆

v) (PLU) 若要消元 w/ pivot 有 $PA = LU$ P46.1

例1. $A = \begin{pmatrix} 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \end{pmatrix}$ max |row| 13, 18, 6, 12 scale factor

1-1 ÷ scale vec 3/13, 6/18, 6/6, 12/12

$P_1 = \begin{pmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 6 & -2 & 2 & 4 \\ -6 & 4 & 1 & -18 \\ 3 & -13 & 9 & 3 \\ 12 & -8 & 6 & 10 \end{pmatrix}$ $E_1 = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ -1/2 & & 1 & \\ -2 & & & 1 \end{pmatrix}$ $\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & 2 & 3 & -14 \\ 0 & -12 & 8 & 1 \\ 0 & -4 & 2 & 2 \end{pmatrix}$ 18, 13, 12

scale factor 保持 初始 val

有 2/18, 12/13, 4/12 3rd max

$P_2 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 6 & -2 & 2 & 4 \\ & -12 & 8 & 1 \\ & 2 & 3 & -14 \\ & 4 & 2 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & 1/6 & 1 & \\ & -1/3 & & 1 \end{pmatrix}$ $\begin{pmatrix} 6 & -2 & 2 & 4 \\ & -12 & 8 & 1 \\ & 0 & 13/3 & -83/6 \\ & 0 & -2/3 & 5/3 \end{pmatrix}$ 18, 12

3rd max, 故 $P_3 = I$

$E_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 2/13 & 1 \end{pmatrix}$

$E_3 P_3 E_2 P_2 E_1 P_1 A = U = \begin{pmatrix} 6 & -2 & 2 & 4 \\ & -12 & 8 & 1 \\ & & 13/3 & -83/6 \\ & & & -6/13 \end{pmatrix}$

NOTE

swap along 消元 同 swap first (permute) 再消元.

即 find new elementary op $\tilde{E}$ st. $\tilde{E}_3 \tilde{E}_2 \tilde{E}_1 P_3 P_2 P_1 A = U$, 有

undo $P_2 P_3$ (只剩 $P_1 A$) 再用 E_1, 再套用回 P_{23} P_{Sq}, P1 c)

$$\begin{cases} \tilde{E}_1 = P_3 P_2 E_1 P_2^{-1} P_3^{-1} = P_3 P_2 E_1 P_2^T P_3^T \\ \tilde{E}_2 = P_3 E_2 P_3^T \\ \tilde{E}_3 = E_3 \end{cases}$$

∴ PLU 有
$$\begin{cases} P = P_3 P_2 P_1 \\ L = \tilde{E}_1^{-1} \tilde{E}_2^{-1} \tilde{E}_3^{-1} \\ U = \tilde{E}_3 \tilde{E}_2 \tilde{E}_1 PA \end{cases}$$

例2 LU on $\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & \cdots \\ & 3 & \\ & -9 & \\ & 12 & \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ & 3 & 1 & 2 & -3 \\ & & & 2 & 1 \\ & & & 4 & 7 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ & 3 & 1 & 2 & -3 \\ & & & 2 & 1 \\ & & & & 5 \end{pmatrix} = U$

化为对角 = 1 $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 3 & 4 & 2 & 1 \end{pmatrix}$

例3 LU 快速解 A^{-1}; find $Ax = e_1$ … $Ax = e_n$ 即化为解 sys eq

* SGD - UV decompose, 令 $A = UV$, SGD $\min \|UV - A\|_2^2$ when U, V learned param

应用 Leontief Input-Output Model

production level (output)

1. econ divid in n sectors that produce goods & services $\vec{x}$ in $\mathbb{R}^n$ production vector

open sector consume goods & services, $\vec{d}$ final demand vector, goods & services demands by open sector

producer 为了生产, 需要 input, intermediate demand

Leontief 想找到 $\vec{x}$ s.t. intermediate demand + $\vec{d}$ = $\vec{x}$, 完全平衡

为此, 假设 for each sector, ∃ unit consumption vector ($\vec{c}$) in $\mathbb{R}^n$, input needed per unit of output

可知 intermediate demand $= x_1\vec{c}_1 + x_2\vec{c}_2 + \dots = C\vec{x}$, $C = [\vec{c}_1\ \vec{c}_2\ \dots]$, $\vec{c}_1 x_1$ 为 sector 1 为了生产 x_1 unit 单位需要消耗的 input

这样 $\vec{x} = C\vec{x} + \vec{d} \Rightarrow (I-C)\vec{x} = \vec{d}$

2. 例如 econ 有 manufacturing, Agriculture, Services 三个 sectors.

Purchased from	Inputs Consumed per Unit of Output M	A	S
M	.50	.40	.20
A	.20	.30	.10
S	.10	.10	.30

如 M 需生产 100 units, 需要 $100\vec{c}_1 = 100\begin{bmatrix}.50\\.20\\.10\end{bmatrix} = \begin{bmatrix}50\\20\\10\end{bmatrix}$

而 $C = \begin{bmatrix}.50 & .40 & .20\\ .20 & .30 & .10\\ .10 & .10 & .30\end{bmatrix}$

如 $\vec{d}$ 为 50 M, 30 A, 20 S, $\vec{x}$ 可用以下找到

$I - C = \begin{bmatrix}.5 & -.4 & -.2\\ -.2 & .7 & -.1\\ -.1 & -.1 & .7\end{bmatrix}$ $\quad [I-C\ \ \vec{d}] \sim [I\ \ \vec{x}]$ $\quad \vec{x} = \begin{bmatrix}226\\119\\78\end{bmatrix}$, 生产水平

3. 实际中 $\vec{x}$ 必须 economically feasible, 即无负数 entries; col sums of $C < 1$, 因 sector 应 input < output

则如 C, $\vec{d}$ 无 negative entries 且 col sums of $C < 1$, $I - C$ 有逆.

比如 $\vec{d}$ 确定, industries 使 $\vec{x}$ 生产水平正好为 $\vec{d}$, 即生产出的正好被全消耗, 则

intermediate demand 为 $C\vec{d}$, 而为了 $C\vec{d}$, 需另 intermediate demand $C(C\vec{d})$ ...

$\vec{x} = \vec{d} + C\vec{d} + C^2\vec{d} \dots = (I + C + C^2 + \dots)\vec{d}$, 根据 $(I-C)(I+C+C^2+\dots+C^m) = I - C^{m+1}$ ①

If col sum of $C < 1$, $C^m \to 0$ ∴ $I - C^{m+1} \to I$

因此 由① $(I-C)^{-1} = I + C + C^2 + \dots + C^m$

4. $\vec{d} \to \vec{d} + \Delta\vec{d}$, $\vec{x} \to \vec{x} + \Delta\vec{x}$, 即 production level change with demand ($\vec{d} + \Delta\vec{d} = (I-C)\vec{x} + (I-C)\Delta\vec{x}$

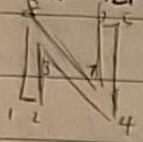
而如 $\Delta\vec{d}$ 只有一个 sector 提高需求, $[I-C]^{-1}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}(I-C)^{-1}\text{的}\\ \text{第一列}\end{bmatrix} = \vec{x}$, 可见哪怕只一个 sector 提高, 整个 sector 都须提高生产力

~~***~~7

note此为T: $\mathbb{R}^3\to\mathbb{R}^3$，但有很好的性质，对满足(章)的$\mathbb{R}^3$ ele，有平移的map，且我们也只关心其余的$\mathbb{R}^2$

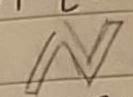
Date. /

应用 图像学

1. 

$D=\begin{bmatrix}0&.5&.5&6&6&5.5&5.5&0\\0&0&6.12&0&8&8&1.58&8\end{bmatrix}$ 被 $A=\begin{bmatrix}1&.25\\0&1\end{bmatrix}$ shear transform

(各个点的坐标)

后为 

2. 但是 matrix multiplication也即 linear Transformation 不能直接平移，因此引入 homogeneous coordinate,
(x,y) in $\mathbb{R}^2$ 可视为 $(x,y,1)$ in $\mathbb{R}^3$，这样 $(x,y)\mapsto(x+h,y+k)$ 可用 $(x,y,1)\to(x+h,y+k,1)$

standard matrix for L.T. 为 $\begin{bmatrix}1&0&h\\0&1&k\\0&0&1\end{bmatrix}$ $\left(\begin{bmatrix}1&0&h\\0&1&k\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix}=\begin{bmatrix}x+h\\y+k\\1\end{bmatrix}\right)$

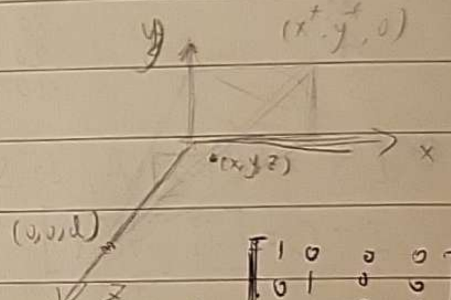
而非平移，即 L.T in $\mathbb{R}^2$ 在 $\mathbb{R}^3$ 中记为 $\begin{bmatrix}A&0\\0&1\end{bmatrix}$，A为 $\mathbb{R}^2$ 的 L.T. 的 standard matrix

如 $\begin{bmatrix}-\cos\varphi&-\sin\varphi&0\\\sin\varphi&\cos\varphi&0\\0&0&1\end{bmatrix}$ counterclockwise rotation $\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$ reflexion $y=x$ $\begin{bmatrix}s&0&0\\0&t&0\\0&0&1\end{bmatrix}$ Scale x by s, y 为 t

3. 3D 同理，可写为 (X,Y,Z,H) if $H\neq0$, and $x=\frac{X}{H}$, $y=\frac{Y}{H}$, $z=\frac{Z}{H}$，为 homogeneous for (x,y,z)

eg. $(10,-6,14,2)$ 与 $(-15,9,-21,-3)$ 都为 $(5,-3,7)$ 的 homogeneous

4. 3D的投影到2D平面



从 $(0,0,d)$ 看，将 (x,y,z) 投到 viewing plane $(x^*,y^*,0)$

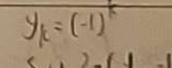
可知 $\frac{x^*}{d}=\frac{x}{d-z}$，$x^*=\frac{x}{1-\frac{z}{d}}$，同理 $y^*=\frac{y}{1-\frac{z}{d}}$

即 $(x,y,z,1)\mapsto\left(\frac{x}{1-z/d},\frac{y}{1-z/d},0,1\right)=(x,y,0,1-z/d)$ $A=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&-1/d&1\end{bmatrix}$ 已

即 ~~[illegible]~~ $\{(x,y,0,t)\}\oplus\{(0,0,z,0)\}=\mathbb{R}^4$，$T:\mathbb{R}^4\to\mathbb{R}^4$ [illegible] $((x,y,z,1)\to(x,y,0,1-z/d))$

而此应用了z一行 [illegible] $(x\,y\,z\,1)$ 而言的

应用 difference equation

1. vector ~~[illegible]~~ space $\mathbb{S}$ of discrete-time signal 中有 signal, function sampled at discrete time intervals
为 form $(y_0,y_1,\dots)$ 每个 y_i 为采样，记为 $\{y_k\}$  $y_k=(-1)^k$ $\{y_k\}=(1,-1,1,-1\ \cdots)$

L.ind for 3 signals 为 $c_1u_k+c_2v_k+c_3w_k=0$ for all k 有 $c_1=c_2=c_3$，k 可为 $\forall\geqslant0$

无论是否 L.ind，上式 k 也可写为 $k+1$, $k+2$，则 $\begin{bmatrix}u_k&v_k&w_k\\u_{k+1}&v_{k+1}&w_{k+1}\\u_{k+2}&v_{k+2}&w_{k+2}\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

左 matrix 为 Casorati matrix of signal, det 为 Casorotian

If Casorati matrix 可逆 for at least one value of k, 则 $c_1=c_2=c_3=0$, $\{u_k\},\{v_k\},\{w_k\}$ L.ind

2. given signal $\{z_k\}$, $a_0y_{k+n}+a_1y_{k+n-1}+\dots+a_{n-1}y_{k+1}+a_ny_k=z_k$ for all k
为 linear difference / linear recurrence equation of order n. If $\{z_k\}$ zero sequence, homogeneous
在 digital signal processing, linear difference equation为 linear filter, $a_0,\dots,a_n$ 为 filter coefficients
$\{y_k\}$ input, $\{z_k\}$ output, $\{z_k\}$为0 则将被 filtered out, $\{y_k\}$ 即为 sol

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3. Sol for homogeneous difference equation has form $y_k = r^k$ for some r.

eg. Find $y_{k+3} - 2y_{k+2} - 5y_{k+1} + 6y_k = 0$ ∀ k.

代入 r^k $r^k(r-1)(r+2)(r-3)=0$. $1^k, (-2)^k, 3^k$ 都为 sol. 比如 3^k,

$3^{k+3} - 2\cdot 3^{k+2} - 5\cdot 3^{k+1} + 6\cdot 3^k$

$= 3^k(27-18-15+6)=0$

综上, r^k 为 sol iff r is root of auxiliary equation $r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n = 0$

4. given $a_1 \dots a_n$, mapping $T: \mathbb{S} \to \mathbb{S}$, $\{y_k\} \mapsto \{w_k\}$ by $w_k = y_{k+n} + a_1 y_{k+n-1} + \dots + a_{n-1} y_{k+1} + a_n y_k$

可证 T linear, 则 sol for homogeneous ($\{w_k\} = \{0\}$) 即 T 的 kernal

1° If $a_n \neq 0$, and $\{z_k\}$ given, $y_{k+n} + a_1 y_{k+n-1} + \dots + a_{n-1} y_{k+1} + a_n y_k = z_k$ for all k has unique sol if $y_0 \dots y_{n-1}$ specified.

□ define $y_n = z_0 - [a_1 y_{n-1} + \dots + a_{n-1} y_1 + a_n y_0]$, 并逐步迭代 define $y_{n+1}, y_{n+2} \dots$

2° set H of all sols of $y_{k+n} + a_1 y_{k+n-1} + \dots + a_{n-1} y_{k+1} + a_n y_k = 0$ for all k is n-dim Vector

□ H 为 $\mathbb{S}$ subspace 因可记为 T 的 kernal. ∀ $\{y_k\}$ in H, $F: H \to \mathbb{R}^n$, $\{y_k\} \to (y_0, y_1, \dots, y_{n-1})$

可证 F linear. 由 1°, ∃ unique $\{y_k\}$ in H st. $F(\{y_k\}) = (y_0\ y_1, \dots y_{n-1})$, F one to one, onto

F isomorphism, 因 $\dim H = \dim \mathbb{R}^n = n$ 见上上页 最后的例)

(3") find basis for set of all sols for $y_{k+3} - 2y_{k+2} - 5y_{k+1} + 6y_k = 0$ ∀ k

由本页 eg 知 $1^k, (-2)^k, 3^k$ 为解, 且由上页 Cartasion, $k=0$ $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ 可逆, L.ind.

any L.ind of 3 vectors in 3-dim space is basis.

~~***~~ 9.2
basis 3

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例1. 算 $Q_{\beta\to\beta'}$,　i) $v_1' = v_1+3v_2-4v_3$　ii) $v_1 = v_1'+v_2'+3v_3'$

$v_2' = 2v_1-v_2+5v_3$　$v_2 = 2v_1'-v_2'+4v_3'$

$v_3' = 4v_1+5v_2+3v_3$　$v_3 = 3v_1'+5v_3'$

法1

□ i) $Q = [I_V]_{\beta}^{\beta'} = (\underset{\beta'\to\beta}{R}^{-1}) = \begin{pmatrix}1&2&4\\3&-1&5\\-4&5&3\end{pmatrix}^{-1}$　ii) $Q = \begin{pmatrix}1&2&3\\1&-1&0\\3&4&5\end{pmatrix}$

法2 i) $Q = [v_1'\cdots v_n']^{-1}[v_1\cdots v_n]$, 而 $\begin{bmatrix}1&3&-4\\2&-1&5\\4&5&3\end{bmatrix}\begin{bmatrix}v_1\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}v_1'\\\vdots\\v_n'\end{bmatrix} \xLeftrightarrow{转置} [v_1\cdots v_n]\begin{bmatrix}1&3&-4\\&\vdots&\end{bmatrix}^T = [v_1'\cdots v_n']$

ii) $\begin{bmatrix}1&1&3\\2&-1&4\\3&0&5\end{bmatrix}\begin{bmatrix}v_1'\\\vdots\\v_n'\end{bmatrix}=\begin{bmatrix}v_1\\\vdots\\v_n\end{bmatrix}$

证1. β r orthonormal basis for $\mathbb{R}^n$, 证 $\underset{\beta\to r}{Q}$ orthogonal

T_{2c}　B col为β

□ 有 $\underset{\beta\to r}{Q}\ \underset{e\to\beta}{Q} = \underset{e\to r}{Q}\ T$　∴ $Q = TB^{-1}$ 由 P55. P vi)

但 一般 change basis matrix 不 orthogonal

应用3. 令 A_{mn} 秩r, β $\mathbb{R}^n$ basis $= \{\overset{\leftarrow R(A^T)\text{ basis}\rightarrow}{x_{1\cdots r}}\ \underset{\leftarrow N(A)\text{ basis}\rightarrow}{x_{r+1\cdots n}}\}$ ← 同P51. T_a 红1, r $\mathbb{R}^m$ basis $\{\overset{\leftarrow R(A)\text{ basis}\rightarrow}{y_{1\cdots r}}\ \underset{\leftarrow N(A^T)\text{ basis}\rightarrow}{y_{r+1\cdots m}}\}$

则 $[L_A]_{\beta}^{r} = \begin{bmatrix}I_{rr}&0\\0&0\end{bmatrix}$　□ $A_i x_i \in R(A)$ ($i\le r$)　$L_A(x_i)=0$ ($i>r$)

例.7 → L.Comb of $y_{1\sim r}$　则 $\underset{[L_A]_e^e}{A} = \underset{\underset{r\to e}{Q}}{T}\begin{pmatrix}I&0\\0&0\end{pmatrix}\underset{\underset{e\to\beta}{Q}}{B^{-1}}$

若 $rk = r = \min(n,m)$ (Full-Rank) 则 $\begin{cases} m<n & [I\ \ 0]\\ m>n & \begin{bmatrix}I\\0\end{bmatrix}\end{cases}$

eg1. $A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$　$\beta = \{\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}4\\5\\6\end{pmatrix} ⋮ \begin{pmatrix}1\\-2\\1\end{pmatrix}\}$

$r = \{\begin{pmatrix}1\\4\\7\end{pmatrix}\begin{pmatrix}2\\5\\8\end{pmatrix} ⋮ \begin{pmatrix}1\\-2\\1\end{pmatrix}\}$

$[L_A]_{\beta}^{r} = [\begin{pmatrix}14\\32\\50\end{pmatrix}_r, \begin{pmatrix}32\\77\\122\end{pmatrix}_r, \begin{pmatrix}0\\0\\0\end{pmatrix}_r] = \begin{pmatrix}\boxed{\begin{matrix}-2&-2\\8&17\end{matrix}}&\begin{matrix}0\\0\end{matrix}\\\begin{matrix}0&0\end{matrix}&0\end{pmatrix}$　I

eg2. $A=\begin{pmatrix}-1&1\\2&-1\\0&1\end{pmatrix}$　$\beta = \{\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\}$

$r = \{\begin{pmatrix}-1\\2\\0\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix} ⋮ \begin{pmatrix}2\\1\\-1\end{pmatrix}\}$

$[L_A]_{\beta}^{r} = [\begin{pmatrix}-1\\2\\0\end{pmatrix}_r, \begin{pmatrix}1\\-1\\1\end{pmatrix}_r] = \begin{pmatrix}\boxed{\begin{matrix}1&0\\0&1\end{matrix}}\\\begin{matrix}0&0\end{matrix}\end{pmatrix}$　I

证2. 对 orthonormal basis (i.e. $\langle e_i, e_j\rangle = \delta_{ij} = \begin{cases}0 & i\ne j\\1 & i=j\end{cases}$), 对 $x=\sum^n \alpha_i e_i$ i.e. 由basis表示

有 $\alpha_i = \langle x, e_i\rangle$　□ 对 $x=\alpha_1e_1+\cdots+\alpha_ne_n$, 内积 w/ e_1: $\langle x\ e_1\rangle = \alpha_1\langle e_1\ e_1\rangle = \alpha_1$

以此类推

~~***~~10. linear Map 2

1 T_1 $T: V \to W$, V, W 为 vector space, T linear, 则 $Null(T) \leq V$, $Range(T) \leq W$

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□ $Null(T)$: i) $T(0_V) = 0_W$, $\therefore 0_V \in Null(T)$ iii) $T(cx) = cT(x) = c \cdot 0_W = 0_W$

ii) $T(x+y) = T(x)+T(y) = 0_W + 0_W = 0_W$

$Range(T)$: i) $T(0_V) = 0_W$, $\therefore 0_W \in Range(T)$ iii) let $T(\alpha) = x$

ii) let $T(\alpha) = x$, $T(\beta) = y$, note $T(\alpha+\beta) = x+y$. $\therefore x+y \in Range(T)$ $\wedge$ 存在 $T(c\alpha) = cT(\alpha) = cx \in Range(T)$

T_2 linear $T: V \to W$, $\beta = \{v_1, \dots, v_n\}$ 为 V basis, 则 $\underset{①}{Range(T)} = \underset{②}{span(T(\beta))} = span(T(v_1), \dots, T(v_n))$

□ 对②: note $T(\beta)$ 即 map 为 $\{T(v_i)\}$. 对① $span(T(\beta)) \subseteq Range(T)$, 因 $T(\beta) \in W$, 且 $\overset{T_1}{Range(T) \leq W}$, 由 T_1

$Range(T) \subseteq span(T(\beta))$, let $\alpha \in V$, $T(\alpha) = x \in Range(T)$, $\overset{T_1}{\alpha = \sum \beta_i v_i}$, T linear, $\therefore x = \sum \beta_i T(v_i)$

例1 对 $T: P_2[\mathbb{R}] \to M_{2\times2}(\mathbb{R})$, $f(x) \to \begin{bmatrix} f(1)-f(2) & 0 \\ 0 & f(0) \end{bmatrix}$, 求 $Range(T)$ basis

□ $P_2[\mathbb{R}]$ 的 basis $\beta = \{1, x, x^2\}$, $Range(T) = span\{T(1), T(x), T(x^2)\} \Rightarrow \{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & \\ & 0 \end{bmatrix}, \begin{bmatrix} -3 & \\ & 0 \end{bmatrix}\} \Rightarrow$ 1st & 2nd
dim = 2

例2 对 $T: P_2[\mathbb{R}] \to P_3[\mathbb{R}]$, $f(x) \to 2f'(x) + \int_0^x 3f(t)dt$, 是否 bijection 四.3 回 T_1

□ 对 $\beta = \{1, x, x^2\}$, $Range(T) = span(\{3x, 2+\frac{3}{2}x^2, 4x+x^3\})$, dim = 3, nullity = 0, $\therefore$ 1-1, 而 $3 < dim(P_3[\mathbb{R}]) = 4$, not onto

T_3 对 $\beta_V = \{v_1, \dots, v_n\}$, 对 $\forall w_1, \dots, w_n \in W$, $\exists$ unique linear $T: V \to W$, such that $\forall v_i \to w_i$

选的不同 w_i 找出不同 T, 这样找到所有 T (即是说, T, U 只需给有 $U(v_i) = T(v_i)$ 则 $U = T$)

□ 对 $x \in V$, $x = \sum^n a_i v_i$, DEF $T: V \to W$, $x \to \sum^n a_i w_i$.

T linear: $T(k\alpha+\beta) = T(\sum^n (kc_{1i}+c_{2i})v_i) = \sum^n (kc_{1i}+c_{2i})w_i$ 而 $T(k\alpha)+T(\beta) = \sum^n kc_{1i}w_i + \sum^n c_{2i}w_i$

且 对 $v_i = \sum_{j\neq i}^n 0 \cdot v_j + v_i$, $T(v_i) = w_i$ 且 unique: 若 $\exists$ $U: V \to W$, 对 $x \in V$, $U(x) = T(x)$ $\therefore U = T$

(但只有 bijection T 不一定是 iso, 见四.2 但也未必不行, 如 $T: \mathbb{R}^2 \to \mathbb{R}^2$, $x \to 0$)

回例2的证法

P_1 i) 对 linear $T: V \to W$, $dim V < dim W$, T 不可能 onto (这即说: T bijection $\Rightarrow dim V = dim W$)

$dim V > dim W$, T 不可能 1-1

□ 由 basis Thm, $rank T \leq dim V < dim W$; $nullity(T) = dim V - rank T \geq dim V - dim V = 0$

ii) T 1-1 $\Leftrightarrow$ T 将 V 的 L.Ind subset map $\to$ W 的 L.Ind subset 1-1, $Null(T) = \{0\}$

□ $\Rightarrow$ 对 V 的 $\beta = \{v_1, \dots, v_n\}$, 考虑 $T(v_i)$, $\sum a_i T(v_i) = 0$, $T(\sum a_i v_i) = 0$, $\sum a_i v_i = 0$, $a_i = 0$

$\Leftarrow$ 法1: 假设不1-1, 则 $\exists$ $x \neq y$ 而 $T(x) = T(y)$. 对 $z = x-y \neq 0$, $\{z\}$ L.Ind. 但 $T(z) = T(x) - T(y) = 0$, $\{0\}$ L.dep. 矛盾

法2, 考虑 $x \in V$ 使 $T(x) = 0$, $\because \{0\}$ L.dep, $\{x\}$ L.dep, 但对单元素的 set L.dep 只有 $x = 0$, $\therefore Null(T) = \{0\}$ 1-1

iii) T 1-1, 且 $S \subseteq V$. S L.Ind $\Leftrightarrow$ $T(S)$ L.Ind.

□ $\Rightarrow$ 由 P_1 ii) 和 T 1-1 $\Leftarrow$ 设 $S = \{s_1, \dots, s_n\}$, 令 $\sum a_i s_i = 0$, $\overset{T \text{ linear}}{T(\sum a_i s_i) = 0}$, $\sum a_i T(s_i) = 0$, $\overset{T(S) \text{ L.Ind}}{a_i = 0}$, S L.Ind.

iv) T 1-1 且 onto, 则 T 将 $\beta_V \to \beta_W$ (反过来也可以, 因此代表 $dim V = dim W$, $V \cong W$ (四.2) $\therefore$ 有 bijection T, 而由 不, 此 bijection unique)

□ 由 P_1 ii) $\{T(v_i)\}$ L.Ind, 且 $Range(T) = span(\{T(v_i)\})$. 而 T onto, $\therefore Range(T) = W$, $\therefore \{T(v_i)\}$ span. $\therefore \{T(v_i)\} = \beta_W$

v) 若 $S \subseteq V$ L.Ind, 而 $T(S)$ L.dep 则 T 不 1-1

□ 考虑若 T 1-1, 则由 P_1 iii) 知 $T(S)$ L.Ind (或: 对 $\sum a_i T(s_i) = 0$, $\exists a_i \neq 0$, 而由 linearity $T(\sum a_i s_i) = 0$

(与题矛 (contrapositive 再 apply iii) 即可)) 因 $\exists a_i \neq 0$, $\therefore \sum a_i s_i \neq 0$, $\therefore Null(T) \neq \{0\}$, not 1-1

vi) 对已知的 β basis for V, given 任意可逆 $n \times n$ 矩阵 Q, 可找出另一 basis r

即: 令 $r_j = \sum_{i=1}^n Q_{ij} \beta_i$, $r = \{r_j\}$ 为 V basis, 且 Q 为 change basis matrix

□ $\sum_j a_j r_j = 0 = \sum_j a_j \sum_i Q_{ij} \beta_i = \sum_i (\sum_j a_j Q_{ij}) \beta_i = 0$ 但 β basis, $\therefore \forall i$, $\sum_j a_j Q_{ij} = 0$

即 $(a_1 \cdots a_n) \begin{pmatrix} Q_{11} & \cdots & Q_{1n} \\ \vdots & \ddots & \vdots \\ Q_{n1} & \cdots & Q_{nn} \end{pmatrix} = (a_1 \cdots a_n) Q = (0 \cdots 0) = 0$, Q 可逆, $(a_1 \cdots a_n) Q Q^{-1} = 0 Q^{-1} = 0$

$\therefore$ 各 $a_j = 0$ $\therefore r$ basis

法3. 假设不1-1, $\exists x \neq 0$ s.t. $T(x) = 0$. 考虑 L.Ind $\beta = \{v_1, \dots, v_n\}$, $x = \sum a_i v_i$, $T(x) = 0 = \sum a_i T(v_i)$, 但对 β 的 $T(\beta)$ 也是 L.Ind.

$\therefore a_i = 0$, 但 $x \neq 0$ 矛盾

***11 linear Maps

例1 $T: M_{2\times3}(F)\to M_2(F)$, $\begin{pmatrix} a_{11} & & \\ & & a_{23}\end{pmatrix}\to\begin{pmatrix} 2a_{11}-a_{12} & a_{13}+2a_{12} \\ 0 & 0\end{pmatrix}$, T bijection?

□ Nullity: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}\end{bmatrix}=\begin{bmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23}\end{bmatrix}=a_{11}\begin{bmatrix}1&2&-4\\0&0&0\end{bmatrix}+a_{21}\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}+a_{22}\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}+a_{23}\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}$

nullity = 4, rank = 2, 不1-1 也不 onto。

应用1 DEF: 对 $W_1\oplus W_2=V$, $T: V\to V$ 为 projection W_1 along W_2 if. $\forall x_1\in W_1, x_2\in W_2$, 对 $x=x_1+x_2$, $T(x)=x_1$

i) ~~对~~ $T:\mathbb{R}^2\to\mathbb{R}^2$, 若T为 project on y-axis along x-axis, $T(a,b)=(0,b)$

project on y-axis along line $L=\{(s,s)\}$, $T(a,b)=T((0,b-a)+(a,a))=(0,b-a)$

ii) 对 $T:\mathbb{R}^3\to\mathbb{R}^3$, 若T为project on xy-plane along line $L=\{(a,0,a)\}$, $T(a,b,c)=T((a-c,b,0)+(c,0,c))$
$=(a-c,b,0)$

iii) T linear 且 $W_1=\{x\in V\mid T(x)=x\}$, $W_1=Range(T)$, $W_2=Null(T)$

□ 对 $x=x_1+x_2$, $y=y_1+y_2$, $T(\alpha x+y)=\alpha x_1+y_1=T(\alpha x)+T(y)$; $W_1\subseteq\{x\in V\mid T(x)=x\}$ trivial ; $x\in$ RHS, $x=x_1+x_2$, $T(x)=x=x_1\in W_1$; $x\in W_1$, $T(x)=x$, $x\in Range(T)$; $x\in Range(T)$, $\exists y=y_1+y_2\in V$ s.t. $T(y)=x=y_1\in W_1$

$x\in W_2$, $T(x)=T(0+x)=0$, $x\in Null(T)$

$x\in Null(T)$, $T(x_1+x_2)=T(x)=x_1=0$, ∴ $x=x_2\in W_2$

iv) 证 ∃ subspace W' and $T: V\to V$ s.t. T project on W along W' 习题 P11 T_3 ***3.R)

□ 令 $\beta_W=\{v_1,\dots,v_k\}$ 为 W basis, 可 extend 为 $\beta_V=\{v_1,\dots,v_n\}$, 也 ~~补~~ 令 $W'=\{v_{k+1},\dots,v_n\}$, $V=W_1\oplus W_2$

应用2 DEF: T $V\to V$ linear, 对 $W\leq V$, W T-invariant if $T(W)\subseteq W$

eg. $\{0\}$, V, Range(T), Null(T) 都 T-invariant   DEF. $T_W: W\to W$ 为 restriction of T on W, $x\to T(x)$

i) 若 W T-invariant, T_W linear □ $T_W(x+ky)=T(x+ky)=T(x)+kT(y)=T_W(x)+kT_W(y)$

ii) 若 T project on W along W', 则 W T-invariant, $T_W=I_W$

□ $T(x+x')=x\in W$, ∴ $T(W)\subseteq W$, 而 $\forall x\in W$, $T_W(x)=T(x)=T(x+0)=x$, $I_W(x)=x$, ∴ $I_W=T_W$

iii) $V=Range(T)\oplus W$ 且 W T-invariant, 则当 V finite-dim, $W=Null(T)$

□ $W\subseteq Null(T)$: 对 $w\in W$, $T(w)\in W$, 但也 $\in Range(T)$, 而 $Range(T)\cap W=\{0\}$, ∴ $T(w)=0$, ∴ $w\in Null(T)$

$Null(T)\subseteq W$: $\dim V=rank T+nullity T=rank T+\dim W$ P17 T_{2b} ∴ 而 $W\subseteq Null(T)$ ⊕ def ***3.T_3, ∴ $W=Null(T)$

iv) W T-invariant, $Null(T_W)=Null(T)\cap W$ 且 $Range(T_W)=T(W)$

□ LHS ⊆ RHS, $x\in W$, $T_W(x)=T(x)=0$, ∴ $x\in$ RHS. ; $x\in Range(T_W)$, $T_W(x')=T(x')=x$, ∴ $x\in T(W)$

RHS ⊆ LHS, $x\in W$ 且 $x\in Null(T)$, 则 $T(x)=0=T_W(x)$; $x\in T(W)$, $T(x')=x$. 而 $T_W(x')=T(x')$

NOTE iii) V 非finite 不成立. eg. $T: P[\mathbb{R}]\to P[\mathbb{R}]$, $f\to f'$, 令 $W=\{0\}$, $Range(T)=P[\mathbb{R}]$, $Null(T)=\{f'为0, f为constant\}$
$=\mathbb{R}\neq W$

例4. V finite dim, $T: V\to V$ linear. if $rank(T)=rank(T^2)$, (1) $Range(T)\cap Null(T)=\{0\}$, 且 $V=Range(T)\oplus Null(T)$

□ 对 $x\in Range(T^2)$, $\exists w\in V$ s.t. $T^2(w)=x$, ∴ 有 $T(w)$ s.t. $T(T(w))=x$, $x\in Range(T)$, ∴ $Range(T^2)\subseteq Range(T)$, 而 rank 相同, ∴ $Range(T^2)=Range(T)$ ***3.T_3, 由rank Thm, $nullity(T)=nullity(T^2)$ 若 $x\in R(T)\cap N(T)$, $T(x)=0$ 且 $\exists w$ st. $T(w)=x$, 则 $T(T(w))=T(x)=0$, ∴ $w\in Null(T^2)=Null(T)$, ∴ $T(w)=0=x$ ∴ x 只可为 0. 而 Range(T), Null(T) ⊆ V, $V=Range(T)+Null(T)$

例5 对 linear $T: V\to V$, 且 $T=T^2$, 证 $V=\{y: T(y)=y\}\oplus Null(T)$

□ $V=\{y\mid T(y)=y\}+Null(T)$: 对 $\forall x\in V$, $x=T(x)+x-T(x)$, $T(x-T(x))=T(x)-T(x)=0$, ∴ $\in Null(T)$; 对 $T(x)$, $TT(x)=T^2(x)=T(x)$

而 intersection $=\{0\}$: 若 $x\neq0$, $T(x)=x=0$, 矛盾

例6. T, U nonzero L.T: $V\to W$, if $R(T)\cap R(U)=\{0\}$, $\{T,U\}$ L.ind subset of $\mathcal{L}(V,W)$

□ $aT+bU=\underline{0}$ 为 0 的 func. 即: 对 $x\neq0\in V$, $aT(x)+bU(x)=0$, $T(ax)=U(-bx)$ ∴ $ax=-bx=0$, $a=b=0$

例7 若 $k\geq n+1$ 则 $\{a_1\dots a_k\}\subseteq\mathbb{R}^n$ L.dep 习.T#3, 而若 $k\geq n+2$, ∃scalar $\alpha_1\dots\alpha_k$ st. 至少1个 $\alpha_i\neq0$, $\sum_{i=1}^k\alpha_i a_i=0$ 且 $\sum^k\alpha_i=0$

□ 设 $\bar a_i=\begin{bmatrix}1\\a_i\end{bmatrix}\in\mathbb{R}^{n+1}$, $\{\bar a_i\}_i$ L.dep ∴ 有非全0 α_i st. $0=\sum^k\alpha_i\bar a_i=\begin{bmatrix}\sum\alpha_i\\ \sum\alpha_i a_i\end{bmatrix}$

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Date. 迁移 回.Ex1的12, 回的col组合

$\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\3\end{pmatrix}+b\begin{pmatrix}2\\4\end{pmatrix}$

1. 17的了, 当B col vector, $B=(b_1,\dots,b_n)^T$, $AB=\sum_{i=1}^{n} b_i A_{:,i}$ 回 P_1 i)&ii) 回.3.ii)

□ ∵ $\{e_1 \dots e_n\}$ 为 $\mathbb{F}^n$ 的basis, $B=\sum_1^n b_i e_i$, ∴ $A(b_1e_1+\dots+b_ne_n) = \underline{b_1Ae_1+\dots+b_nAe_n} = \underline{\sum b_i A_{:,i}}$

类似的, 当A row vector, $A=(a_1 \dots a_n)$ $AB=\sum_{i=1}^{n} a_i B_{i,:}$ □ $AB=(B^TA^T)^T$ → 转化上行情况

$(a\ b)\begin{pmatrix}1&2\\3&4\end{pmatrix} = a(1\ 2)+b(3\ 4)$

$\dim = n \quad m \quad A=[T]_\beta^\gamma$

2. $T:V\to W$, 证 $\text{rank}(T)=\text{rank}(L_A)$ $\text{nullity}(T)=\text{nullity}(L_A)$

$$\begin{tikzcd}V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] \\ \mathbb{F}^n \arrow[r, "L_A"] & \mathbb{F}^m\end{tikzcd}$$

□ $\text{rank}\, T = \dim(\underline{R(T)})$ $\leq W$ $\text{rank}\, L_A = \dim \underline{R(L_A)}$ $\leq \mathbb{F}^m$

由回例1想到 $\phi_\gamma: W \to \mathbb{F}^m$, WTS: $\phi_\gamma: R(T) \to \boxed{\phi_\gamma R(T) = R(L_A)}$

⊆): $x'' \in \phi_\gamma R(T)$, 有 $x' \in R(T)$ st. $\phi_\gamma(x')=x''$, 有 $x\in V$ st. $\phi_\gamma(T(x))=x''$ ∵ $L_A\phi_\beta(x)=x''$ 回.2

∴ 有 $\phi_\beta(x)$ st. $L_A(\phi_\beta(x))=x''$ ⊇) 类似, ∴ $\phi_\gamma R(T) = R(L_A)$

∴ 由回例1. $\text{rank}\, L_A = \underline{\dim R(L_A) = \dim(\phi_\gamma R(T))} = \underline{\dim R(T)} = \text{rank}\, T$

nullity 由 rank Thm 知

3. 对AB m rows, $N(A^T) \subseteq N(B^T) \Rightarrow R(B) \subseteq R(A)$ 相同, 因若 $u\in N(A^T)$, 由⊆ $u\in N(B^T)$

□ 考虑 $[A\ v]$ (多1 col), 由回.3, $\begin{cases}\dim R(A) + \dim N(A^T) = m \\ \dim R([A\ v]) + \dim N([A\ v]^T) = m\end{cases}$ $a^Tv = u^TBx = (B^Tu)^Tx = 0$ 而 $[A\ v]^Tu = \begin{bmatrix}A^Tu\\v^Tu\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

令 $v\in R(B)$, 有 x st. $Bx=v$

$\Rightarrow \dim R(A) = \dim R([A\ v])$ ∴ v 可由A的col L.comb. ∴ $v\in R(A)$ 得证

4. 对 A_{mn} B_{np} $\dim R(AB) = \dim R(B) - \dim(N(A)\cap R(B))$

□ 令 map $T: R(B) \to R(AB)$, $y \to Ay$. 因 $T(cy_1+y_2) \overset{Bx_1=y_1}{=} T(B(cx_1+x_2)) = AB(cx_1+x_2)$

$\text{Null}(T) = N(A)\cap R(B)$ 因 $y\in N(T) \Leftrightarrow Ay=0$ 且 $y\in R(B)$

$= \underbrace{cABx_1}_{cT(y_1)} + \underbrace{ABx_2}_{T(y_2)}$, linear. ∴ 由回.3, $\dim R(B) = \text{nullity}\, T + \underbrace{\text{rk}\, T}_{\dim R(AB)}$

5. $N(A)=N(A^TA)$ □ ⊆ 因 $Ax=0$ A^TAx 必 $=0$ ⊇ 有 $A^TAx=0$ $x^TA^TAx = 0 = \|Ax\|^2$, 由性质0 Ax 必 $=0$

6. $V \leq \mathbb{R}^n$, 证 ∃ matrix U V st. $V=R(V)=N(U)$ Ps1. $V \oplus V^\perp = \mathbb{R}^n$

□ 令 $\{v_1 \dots v_r\}$ basis of dim=r V, $V=[v_1 \dots v_r]$, $R(V)=V$ 又令 $\{u_1 \dots u_{n-r}\}$ basis of $V^\perp$

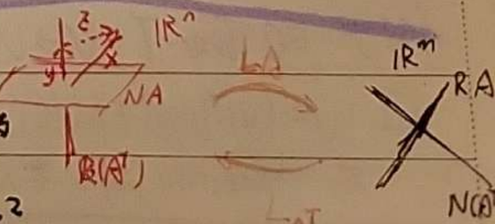
$U = \begin{bmatrix} -u_1- \\ \vdots \\ -u_{n-r}- \end{bmatrix}$, $V^\perp = R(U^T) \overset{取\perp}{\Longrightarrow} \underline{V = R(U^T)^\perp = N(U)}$ Ps1

7. 证 $\overset{A_{mn}}{L_A}$ bijection $R(A^T) \to R(A)$; L_{A^T} bijection $R(A) \to R(A^T)$

□ 1° onto) $b\in R(A)$, 由 $R(A)$ 定义 有 $Ax=b$, $x\in\mathbb{R}^n$, 但 $x = \underset{R(A^T)}{y} + \underset{NA}{z}$ Ps1. (m 2 A) + 2 (0) ∴ $Ax=Ay=b$.

1-1) $Ax=0$ 只有trivial sol, 因若有 $x\neq 0 \in R(A^T)$, 即有 $A^Ty=x$, $AA^Ty=0$ 只有 $y=0$

2° 由1°, L_{A^T} 为 $R(A) \to R(A^T)$ bijection   见左图1. 7.2



liner Map 4

Date. /

1. 由~~xxx~~2.2. 提出矩阵乘法:

对 $T: V\to W$, $U: W\to Z$, let $A=[U]_\beta^\gamma$, $B=[T]_\alpha^\beta$

$\alpha=\{v_1 \cdots v_n\}$ $\beta=\{w_1 \cdots w_m\}$ $\gamma=\{z_1 \cdots z_p\}$, consider $U\circ T: V\to Z$, 易证其 linear,

DEF $AB=[UT]_\alpha^\gamma$, 对 v_j: $UT(v_j)=U(\sum_{k=1}^m B_{kj}w_k) \overset{定义, [T]_\alpha^\beta}{=} \sum_k^m B_{kj}U(w_k)$ linearity

$=\sum_{k=1}^m B_{kj}(\sum_{i=1}^p A_{ik}z_i)=\sum_{i=1}^p(\sum_{k=1}^m A_{ik}B_{kj})z_i$, ∴ 令 $C_{ij}=\sum_{k=1}^m A_{ik}B_{kj}$, 故 $(AB)_{ij}=C_{ij}$

NOTE, 因 $f\circ g$ 一般 $\neq g\circ f$, 很容易知 $AB\neq BA$, $\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}1&1\\0&0\end{pmatrix}$

易证 $[UT]_\alpha^\gamma=[U]_\beta^\gamma[T]_\alpha^\beta$, 几乎是定义 $\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\1&1\end{pmatrix}$

2. P_1: 正如 对 $U_1, U_2\in\mathcal{L}(V,W)$, $T\in\mathcal{L}(W,Z)$, $T(U_1+U_2)=T(U_1)+T(U_2)$ 等, matrix representation 有 (一些性质是它们的例子)

let A m×n, B.C n×p, D E q×n

i) $A(B+C)=AB+AC$; $(D+E)A=DA+EA$ ☐ $[A(B+C)]_{ij}=\sum_{k=1}^n A_{ik}(B+C)_{kj}=\sum A_{ik}B_{kj}+\sum A_{ik}C_{kj}$

ii) 对 $k\in F$, $k(AB)=(kA)B=A(kB)$

Kronecker delta, $\delta_{ij}=\begin{cases}1, & i=j\\0, & i\neq j\end{cases}$, 之后 I_n 可由 δ_{ij} def: $(I_n)_{ij}=\delta_{ij}$

iii) $I_mA=A=AI_n$ ☐ $(I_mA)_{ij}=\sum_{k=1}^m(I_m)_{ik}A_{kj}=\sum_{k=1}^m\delta_{ik}A_{kj}=A_{ij}$ 只有当 $k=i$ $\delta_{ik}=1$

Identity transform: V→V, x→x

iv) if $\dim V=n$, $[I_V]_\beta=I_n$ ☐ $[I_V]_\beta=[[I_V(\beta_1)]_\beta,\dots]=[[\beta_1]_\beta,\dots]=[e_1,\dots,e_n]=I_n$

NOTE cancel law 不成立, eg $A=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $A^2=0=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, 若 $A\cdot A=0=A\cdot 0$, $A=0$, 矛盾

3. 因 T_3 的证明 let A m×n, B n×p, i) $\overset{AB_j (o)}{AB_{:,j}}=A\cdot B_{:,j}$ ii) $B_{:,j}=Be_j$

☐ i) $AB_{:,j}=\begin{pmatrix}(AB)_{1j}\\\vdots\\(AB)_{mj}\end{pmatrix}=\begin{pmatrix}\sum A_{1k}B_{kj}\\\vdots\\\sum^n A_{mk}B_{kj}\end{pmatrix}=A\begin{pmatrix}B_{1j}\\\vdots\\B_{nj}\end{pmatrix}=AB_{:,j}$ ii) $(Be_j)_i=\sum_k^p B_{i,k}e_k=B_{i,j}$ 只有当 $k=j$ $e_k=1$

OR: 左右 $(BI)_{:,j}=B\cdot I_{:,j}=B\cdot e_j$

当然 $AB_{i,:}=A_{i,:}\cdot B$, 但用的少 (

☐ $(AB)_{i,:}=((AB)_{i,1},\dots,(AB)_{i,p})=(\sum_{k=1}^n A_{ik}B_{k,1},\dots,\sum_{k=1}^n A_{ik}B_{kp})=[A_{i1},\dots,A_{in}]B=A_{i,:}B$

4. ~~xxx~~2. 结论的另一证明: $[T(u)]_\gamma=[T]_\beta^\gamma[u]_\beta$

☐ fix $u\in V$, def $f: F\to V$, $a\to au$, let $\alpha=\{1\}$ basis for F. note $g=Tf$

$g: F\to W$; $a\to aT(u)$ 按1的是α"记"

$[T(u)]_\gamma=[g(1)]_\gamma=[g]_\alpha^\gamma=[Tf]_\alpha^\gamma=[T]_\beta^\gamma[f]_\alpha^\beta=[T]_\beta^\gamma\cdot[f(1)]_\beta=[T]_\beta^\gamma[u]_\beta$

ele 以 γ 表示 由定义 $[g]_\alpha^\gamma=[[g(\text{basis})]_\gamma]=[[g(1)]_\gamma]$ F的 basis 只有 $\{1\}$

5. DEF $L_A: F^n\to F^m$, $x\to Ax$, A m×n. L_A linear (由於 P_1 i)&ii)) 据3.ii)

P_2 i) $[L_A]_\beta^\gamma=A$ ☐ $[[L_A(e_1)]_\gamma,\dots,[L_A(e_n)]_\gamma]=[[Ae_i]_\gamma]_{i=1}^n=[A_{:,i}]_{i=1}^n=A$

ii) $L_A=L_B\iff A=B$ ☐ $A=[L_A]_\beta^\gamma=[L_B]_\beta^\gamma=B$, ∴ $A=B$

iii) $L_{A+B}=L_A+L_B$; $L_{kA}=kL_A$ ☐ P_1 i)&ii)

iv) (回 P_1) $T: F^n\to F^m$ liner, ∃ unique m×n C st. $T=L_C$. ☐ $C=[T]_\beta^\gamma$, $[T(x)]_\gamma=[T]_\beta^\gamma[x]_\beta=Cx$ unique 见 P_2 ii)

v) let E n×p, $L_{AE}=L_AL_E$ (此证明矩阵乘 associative, $L_{A(BC)}=L_AL_{BC}=(L_AL_B)L_C$)

☐ $L_{AE}(e_j)=(AE)(e_j)=\underset{据3.ii)}{AE_{:,j}}=\underset{据3 i)}{A\cdot E_{:,j}}=A\underset{据3.ii)}{(E\cdot e_j)}=L_AL_E(e_j)$, ∴ 两者相同 (~~xx~~10, T_3 证了)

例1 $T: V\to W$ isomorphism, $V_0 \le V$, i) 证 $T(V_0)\le W$, ii) $\dim V_0 = \dim T(V_0)$

□ i) 对 $x', y' \in T(V_0)$ 对 $x'+y' = T(\overset{\in V}{x+y}) \in T(V_0)$ ii) T 1-1, ∴ 对 β_{V_0}, $T(\beta_{V_0})$ L.Ind ($\overset{\text{~~xx~~ 1.9 } P_{??}}{}$)

∃$x,y\in V$, $T(x)=x'$, $T(y)=y'$ $kx' = T(kx) \in T(V_0)$ 且 $\forall x'\in T(V_0)$, 有 $T(x)=x'$, 而 $x\in\langle\beta_{V_0}\rangle$

∴ $x' = \sum a_i T(\beta_{V_0,i})$ ∴ span

∴ $\{T(\beta_{V_0})\}$ 为 basis, 因 β_{V_0} 可此推

例2. V basis $\beta=\{v_1\cdots v_n\}$ W basis $\gamma=\{w_1\cdots w_m\}$, $\underline{\exists!\ T_{ij}: V\to W}$, $v_k \to \begin{cases} w_i & k=j \\ 0 & k\ne j\end{cases}$ (~~xxx~~ 1.0. T_3)

证 $\{T_{ij}: i=1\cdots m, j=1\cdots n\}$ basis for $\mathcal{L}(V,W)$, M^{ij} 为 $m\times n$, 全0除了row i col j 为 1, 证

$M^{ij} = [T_{ij}]_\beta^\gamma$, 且 $\underline{\exists!}\ \Phi: \mathcal{L}(V,W)\to M_{m\times n}(\mathbb{F})$, $T_{ij}\to M^{ij}$, 证 Φ isomorphism (~~xx~~ 1.0. T_3)

□ i) $\sum\sum a_{ij}T_{ij}=0$, $\mathcal{L}(V,W)$ 的0为 $f(x)=0$, $\forall x\in V$. ∴ $\overset{\text{特别地}}{\sum_i\sum_j} a_{ij}T_{ij}(v_k)=0$ ∴ $\sum_i a_{ik}T_{ik}(v_k)=0$

对$k=1\cdots n$, $\sum_i a_{ik}w_i \overset{!}{=} 0$ ∴ $a_{ik}=0$ 而 k 又 arbitrary ∴ L.Ind.

ii) $[T_{ij}]_\beta^\gamma = [[T_{ij}(v_k)]_\gamma]_{k=1}^n = M^{ij}$ (只有$k=j$才有w_i, 且只有i row有1 $[w_i]_\gamma = e^i$)

iii) note 由i) T_{ij} basis $\Phi(T_{ij}) = M^{ij}$ 又为 $M_{m\times n}(\mathbb{F})$ 的 basis, ∴ 由 (~~xx~~ 1.0. $P_{??}$) isomorphism

例3 $S: P_3[\mathbb{R}] \to M_2(\mathbb{R})$, $f \to \begin{pmatrix} f(1) & f(2) \\ f(3) & f(4)\end{pmatrix}$ is S invertible?

□ 法1 $S(ax^2+bx+c) = \begin{pmatrix} f(1) & f(2) \\ f(3) & f(4)\end{pmatrix} = 0$ ∴ $x=1,2,3,4$ 时 $f=0$, ∴ $a=b=c=0$, Null$(S)=\{0\}$, 1-1, $\boxed{\text{III}}$ 4: 且 $\dim P_3[\mathbb{R}] = \dim M_2(\mathbb{R})$, $\underline{\text{bijection}}$ invertible

法2 $[S]_\beta^\gamma = \ldots$ 证 $[S]_\beta^\gamma$ 之矩阵可逆

法3 直接找出 $S^{-1}: \begin{pmatrix} a & b \\ c & d\end{pmatrix} \to p$ $\begin{cases} p(1)=a \\ p(2)=b \\ p(3)=c \\ p(4)=d \end{cases} \xrightarrow{\text{lagrange}} p(x) = \sum_{i=0}^{3} p(i+1) f_i$, 证 $S(S^{-1}(x)) = x$

例4. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ let $\beta = \{\binom{1}{2}, \binom{3}{4}\}$ $\gamma = \{\binom{5}{6}, \binom{7}{8}\}$ 找出 $[L_A]_\gamma^\beta = P^{-1}AQ$ 的 P Q

□ 法1 $[L_A]_\gamma^\beta = [I]^\beta [L_A]_{\underset{\text{标准基}=\alpha}{}} [I]_\gamma^\alpha = ([I]_\beta^\alpha)^{-1} \underset{P^{-1}}{A} ([I]_\gamma^\alpha) = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1} A \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}$

法2 $[L_A]_\gamma^\beta = [I]_\gamma^\beta [L_A]_\gamma^\gamma = \underline{[(\tbinom{5}{6})_\beta (\tbinom{7}{8})_\beta]\ \underset{\gamma\to\alpha}{Q^{-1}}}\ A\ \underline{\underset{\gamma\to\alpha}{\overset{Q}{Q}}}$, 即 $P = Q([I]_\gamma^\beta)^{-1} = Q[I]_\beta^\gamma$

In fact, T invertible ⟺ $[T]_\beta^r$ invertible

□ ⇒ 校1. ⇐ let B $n\times n$ st. $B[T]_\beta^r = I_n$，由校2.整本，Φ的$\mathcal{L}(W,V) \cong M_{n\times n}(\mathbb{F})$，和 ※※10.$T_3$，

$\exists!\, U \in \mathcal{L}(W,V)$, st. $U: \underset{\in r}{w_j} \to \sum_{i=1}^{n} B_{ij} \underset{\in \beta}{v_i}$，∴ $B=[U]_r^\beta$，WTS $U=T^{-1}$: $[UT]_\beta = [U]_r^\beta [T]_\beta^r = BA = I_n = [I_V]_\beta$

(note i=j, dim相同)　即$\boxed{T}$.P_2 iv)　∴ $UT = I_V$　$\boxed{T}$.P_1 iv)

1. 对 bijection liner $T: V\to W$，考虑 $T^{-1}\ W\to V$ 也 linear，由此引入矩阵 invertible

A B

根据 T invertible，T^{-1} 也 invertible，$[T^{-1}]_r^\beta = \left([T]_\beta^r\right)^{-1}$

※※10.P_i)　identity map，记A　$\boxed{T}$ P_1 iv)　同.1 notation

□ 因T bijection，$\underline{\dim V = \dim W = n}$，$[T]_\beta^r$ $n\times n$，而 $T^{-1}T = I_V$，∴ $I_n = [I_V]_\beta = [T^{-1}T]_\beta = [T^{-1}]_r^\beta [T]_\beta^r$

同理 $I_n = [I_W]_\beta = [TT^{-1}]_\beta = [T]_\beta^r [T^{-1}]_r^\beta$

∴ invertible

2. 事实上，只要 $\dim V = \dim W \iff V \cong W$（即 ∃ bijection liner $T: V\to W$）

※※10.P_{ii})　※※10.T_2

□ ⇒ 令 $\beta=\{v_1 \cdots v_n\}$，$r=\{w_1 \cdots w_n\}$ 由 ※※10.T_3 ∃ unique $T: V\to W$, $v_i \to w_i$，∴ $\underline{T\ 1\text{-}1}$，而 $\underline{\mathrm{Range}(T) = \mathrm{span}(T(v_i)) = \mathrm{span}(w_i) = W}$

∴ onto ∴ bijection

⇐ ※※10.P_i)

NOTE，由此 当 $\dim V = n$，$V \cong \mathbb{F}^n$，另一种视角看 ※※2.2

1° $\Phi: \mathcal{L}(V,W) \to M_{\dim(W)\times\dim(V)}(F)$, $T \to [T]_\beta^r$　　2° $\phi_\beta: V \to \mathbb{F}^{\dim(V)}: x \to [x]_\beta$　↗ ※※2.2

两者皆 isomorphism

liner 见 ※※2.T

bijection: $\forall A \in M$, ∃且∃! T st. $\Phi(T) = [T]_\beta^r = A$，由$[T]_\beta^r$的定法知 unique

这说明 $\dim(\mathcal{L}(V,W)) = \dim(W)\times\dim(V)$ （$M_{m\times n}(\mathbb{F})$ dim mn）

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] \arrow[rd, bend left] \arrow[rd, bend right] & W \arrow[d, "\phi_r"] \\
\mathbb{F}^n \arrow[r, "L_A"'] & \mathbb{F}^m
\end{tikzcd}$$

对 $x\in V$，$L_A\phi_\beta(x) = [x]_r$，即: $[T]_\beta^r[x]_\beta = [T(x)]_r$（见$\boxed{T}$.4）

对 $x\in V$，$\phi_r T(x) = [x]_r$

∴ $L_A\phi_\beta = \phi_r T$

NOTE 即使 $W\cong V$，不代表 只有一个 isomorphism. 考虑 ※※4.2 basis change

即左相似性是对A ($M_n(\mathbb{F})$)

3. 由 ※※ T_{2b} 当 $T: L_A: \mathbb{F}^n \to \mathbb{F}^n$，$[L_A]_r = Q^{-1}_{r\to\beta}[L_A]_\beta Q_{r\to\beta} = Q^{-1}\underline{A}\,\underline{Q}$ → j col 即 r_j

eg. $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{pmatrix}$ $r = \left\{\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$，$Q = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. ∴ $[L_A]_r = Q^{-1}AQ = \begin{pmatrix} 0 & 2 & 8 \\ -1 & 4 & 6 \\ 0 & -1 & -1 \end{pmatrix}$

(β, γ standard)

例1 1° let $\beta=\{\binom{2}{1}\binom{3}{1}\}$ basis for $\mathbb{R}^2$, 求 β^*, 2° $T: P_1[\mathbb{R}]\to\mathbb{R}^2$, $p(x)\to(p(0),p(2))$ 求 $[T^t]_{\gamma^*}^{\beta^*}$

□ 1° $\beta^*=\{f_1,f_2\}$ 对 f_1 $f_1\binom{2}{1}=1=f_1(2*\binom{1}{0}+\binom{0}{1})=2f_1\binom{1}{0}+f\binom{0}{1}$ $f_1(x,y)=-x+3y$

$f_1\binom{3}{1}=0=3f_1(e_1)+f_1(e_2)$ ∴ $f_1(e_1)=-1$, $f_1(e_2)=3$ $f_2(x+y)=x-2y$

2° $[[g_1T]_{\beta^*},[g_2T]_{\beta^*}]=\begin{bmatrix}a&c\\b&d\end{bmatrix}$

之行 $g_1T=af_1+bf_2$, $g_1T(1)=g_1\binom{1}{1}=g_1(e_1)+\underline{g_1(e_2)}^{=0}=1$; $g_1T(1)=af_1(1)+bf_2(1)=a$ ∴ $a=1$

$g_1T(x)=g_1\binom{0}{2}=2g_1(e_2)=0$; $g_1T(x)=af_1(x)+bf_2(x)=b$ ∴ $b=0$ ∴ $\begin{bmatrix}1&1\\0&2\end{bmatrix}$

例2. $V=\mathbb{R}^3$, $f_1\begin{pmatrix}x\\y\\z\end{pmatrix}=x-2y$, $f_2\begin{pmatrix}x\\y\\z\end{pmatrix}=x+y+z$, $f_3\begin{pmatrix}x\\y\\z\end{pmatrix}=y-3z$, 证 $\{f_1\cdots f_3\}$ basis of V^* 且找出V ~~对~~应的basis

□ i) 设 $\sum a_if_i\begin{pmatrix}x\\y\\z\end{pmatrix}=\underline{\underline{0}}^{\text{zero func}}$ ∴对 $\begin{pmatrix}x\\y\\z\end{pmatrix}$, $\sum a_if_i\begin{pmatrix}x\\y\\z\end{pmatrix}=\underline{\underline{0}}^{\text{任为}0}$ ∴ $\begin{cases}a+b=0\\2a+b+c=0\\b-3c=0\end{cases}$ ⇒ $a=b=c=0$

ii) $f_1(v_1)=1$, $f_2(v_1)=0$, $f_3(v_1)=0$ ⇒ $\begin{cases}x_1-2y_1=1\\x_1+y_1+z_1=0\\y_1-3z_1=0\end{cases}$ ∴ $v_1=(\frac{2}{5},\frac{-3}{10},\frac{-1}{10})$ 同理 $v_2=(\frac{3}{5},\frac{3}{10},\frac{1}{10})$ $v_3=(\frac{1}{5},\frac{1}{10},\frac{-3}{10})$

例3. ∀ basis in V^* 为 V 一些 basis 的 dual basis, 即对 $x=\sum a_i\underset{-}{?_i}^{\text{V的某basis}}$, $f_j(\sum a_i?_i)=a_j$

□. let $\{f_1\cdots f_n\}$ basis for V^*, 将 V^* 视为右边1的PEF 中的V, 总∃ 由 V^* basis 做出的 V^{**} basis $\{\hat{x}_1\cdots\hat{x}_n\}$

由 def 得出, 对于 $t\in V^*$, $t=\sum a_if_i$, $\hat{x}_j(t)=\hat{x}_j(\sum a_if_i)=a_j=t(x_j)$, 特别地, 对于 f_j,

$\{x_i\}$为V basis, 因 ^ isomorphism, 且有一个 $x_i\to\hat{x}_i$, 且个数一样, ∴ basis

$f_j(\underset{=}{x}^{\in V})=f_j(\sum a_i\underset{=}{x_i})=a_j$, ∴ $\{f_1\cdots f_n\}$ 是 $\{x_1\ldots x_n\}$ 的 dual basis

特为linear functional

Linear Map 6

SQL
(aggregate func 也是其中一种)

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rr, bend right, "gT \in V^*"'] & W \arrow[r, "g \in W^*"] & \mathbb{F} \end{tikzcd}$$

Date. /

↑ vec. sp. over field

1 对 L.T : $\underline{\underline{V}} \to \underline{\underline{F}}$，令 $V^* = \mathcal{L}(V, F)$ dual space 正如 ※※2. 2的DEF，$\mathcal{L}(V,F)$为 V.S.

由 19、2 结论，$\dim(V^*) = \dim V \cdot \dim F = \dim V$，∴ $V^* \cong V$ (19.2)

DEF：对 $x \in V$，$x = \sum a_i v_i$，$f_i(x) = a_i$ 为 coord function

T_{1a} 令 $\beta^* = \{f_1 \dots f_n\}$，$\beta^*$ basis for V^*，即：对 $\forall f \in V^*$，$f \in \{f_1 \dots f_n\}$

☐ 令 $g = \sum f(v_i) f_i$，WTS $g = f$ ∴ span：$\underline{g(v_j)} = \sum f(v_i) f_i(v_j) = \sum f(v_i) \delta_{ij} = f(v_j)$

g-f 对basis map 相同 ※※10. T_5　四. P. iii) 的 kronecker delta　※※3. P. d) span. ∴ basis

T_{1b} 对 $T: V \to W$，DEF $T^T: W^* \to V^*$，$g \to gT$，$[T^T]_{r^*}^{\beta^*} = ([T]_\beta^r)^T$ 见上图

☐ i) 对 $g \in W^*$，$T^T(g) = gT \in \mathcal{L}(V, F)$，∴ T^T 将其 map 到 V^*　令 $A = [T]_\beta^r$

ii) $T^T(f + kg) = fT + kgT$，linearity　iii) 对 $\beta = \{v_1 \dots v_n\}$ $r = \{w_1 \dots w_m\}$ $\beta^* = \{f_1 \dots f_n\}$ $r^* = \{g_1 \dots g_m\}$ 考虑 $[T^T]_{r^*\, ij}^{\beta^*} =$

即 $[T^T(g_j)]_{\beta^*}$ 的 i-row，由 T_{1a}，$T^T(g_j) = g_j T \in V^*$，∴ $= \sum_i g_j T(v_i) f_i$ =

※※2 lin 构造

∴ i-row $= g_j T(v_i) = \underline{g_j(\sum_k A_{ki} w_k)}$ (※※2 构造) $= \sum_k A_{ki} g_j(w_k) = \sum_k A_{ki} \delta_{jk} = A_{ji}$

2. 考虑 V^{**}，Def $\hat{x}: V^* \to F$，$f \to f(x)$ $(x \in V)$ 易证 $\hat{x}$ L.T.，证 $\psi: V \to V^{**}$，$x \to \hat{x}$ 为 isomorphism

☐ i) L.T.、考虑 $\widehat{x+ky}$ 与 $\hat{x} + k\hat{y}$，对各 $f \in V^*$，$\widehat{x+ky}(f) = f(x+ky) = f(x) + kf(y)$ (V^* 为 set of L.T.) $= (\hat{x} + k\hat{y})(f)$

ii) 1-1. 若 $\psi(x) = 0$，即 $\hat{x} = \underline{\underline{0}}$ (zero func)，即 对$f \in V^*$，$\hat{x}(f) = \underline{\underline{0}}$ (scalar)，$f(x) = 0$，而 $x \in V$，$x = \sum a_i v_i$，$\sum a_i f(v_i) = 0$

因 f 任意，对各 f_j，$\sum_i a_i f_j(v_i) = 0$，$\sum_i a_i \delta_{ji} = 0$，各 $a_j = 0$ ∴ $x = 0$ (四. T_1)

iii) onto：

∵ $\dim V^{**} = \dim \mathcal{L}(V^*, F) = \dim V^* \cdot \dim F = 1 \cdot \dim V$ ∴ 同dim 又 1-1. ∴ onto (四. 4.)

3. T_{2a} 令 $\underline{W} < V$ (proper)，∃ nonzero linear functional $f \in V^*$ st. $f(x) = 0$，$\forall x \in W$

☐ 对 W basis $\{v_1 \dots v_m\}$ 延伸至 V basis $\{v_1 \dots v_n\}$，$W < V$ ∴ $m < n$. 取 $f_{m+1} \in V^*$ basis，$\forall x \in W$，$f_{m+1}(\sum_1^m a_i v_i) = 0$

※※3. T_3　∵ ∃，由於1. 的 Def

而对 $v_{m+1} \in V$，$f_{m+1}(v_{m+1}) = 1$，∴ f_{m+1} nonzero

T_{2b} 令 $T: V \to W$.　i) T onto ⟺ T^T 1-1
ii) T 1-1 ⟺ T^T onto

☐ i) (⇒) let $g \in \mathrm{Nul}(T^T)$，$g \in W^*$，$\forall w \in W$，consider $g(w) = \underline{g(T(w'))} = \underline{T^T(g)}(w') = \underline{\underline{0}} \in \mathbb{F}$　(onto)　(zero func，因 $g \in \mathrm{Nul}$)

∴ g zero func. ∴ $\mathrm{Nul}(T^T) = \{0\}$ ∴ 1-1.(因 4. 7.)　(⇐) assume $R(T) \neq W$，而 $\underline{R(T) < W}$，由 T_{2a}，

※※10. T_1

有 nonzero $g \in W^*$，且 $g(w) = 0$，$\forall w \in R(T)$. 但 因 nonzero，$T^T(g) \neq \underline{\underline{0}}$ (zero func) 但 $\forall x \in V$，$gT(x) = g(\underset{=}{?}) = 0$ ($\in R(T)$) ∴ 矛盾

ii) ⇒ 因 $R(T) \leq W$，且 $\{T(v_i)\}_i$ 为 R(T) basis (※※10 P. i)) (V' basis)，对 $\forall f \in V^*$，Def $g \in W^*$ as：$gT(v_i) = f(v_i)$ $g(\underline{w_j}) = 0$

W basis 中非 $T(v_i)$ 的

note 这样 $T^T(g)(v_i) = gT(v_i) = f(v_i)$ ∴ $\underline{T^T(g) = f}$ ※※10. T_5. basis map 相同 ∴ onto

⇐ assume 非 1-1，∃ $x \neq 0 \in V$，st. $T(x) = 0$，def $f \in V^*$ as $f(x) \neq 0$，而 T^T onto ∴ 有 $g \in W^*$ st. $gT = f$

consider $gT(x) = g(0) = 0$　但 $f(x) \neq 0$ ∴ 矛盾
($g \in \mathcal{L}(W, \mathbb{F})$，g L.T. send $\underset{\in W}{\underline{\underline{0}}} \to \underset{\in \mathbb{F}}{\underline{\underline{0}}}$)

deli得力

例1. product of 2 matrix, rank 为两 matrix 较小的 rank □ 可能会有 $AB=0$, $A,B\neq 0$ ∴ ×

例2 将 $A=\begin{pmatrix}1&2&1\\1&0&1\\1&1&2\end{pmatrix}$ 写为 elementary 的乘积

□ A 可逆 由超 $P_{iii)}$ $A\sim\begin{pmatrix}1&2&1\\0&-2&0\\1&1&2\end{pmatrix}\sim\begin{pmatrix}1&2&1\\0&-2&0\\0&-1&1\end{pmatrix}\sim\begin{pmatrix}1&2&1\\0&1&0\\0&-1&1\end{pmatrix}\sim\begin{pmatrix}1&0&1\\0&1&0\\0&-1&1\end{pmatrix}\sim\begin{pmatrix}1&&1\\&1&\\&&1\end{pmatrix}\sim\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}$ $(I_n=E_6\cdots E_1A)$

$E_1=\begin{pmatrix}1&&\\-1&1&\\&&1\end{pmatrix}$ $E_2=\begin{pmatrix}1&&\\&1&\\-1&&1\end{pmatrix}$ $E_3=\begin{pmatrix}1&&\\&-1/2&\\&&1\end{pmatrix}$ $E_4=\begin{pmatrix}1&-2&\\&1&\\&&1\end{pmatrix}$ $E_5=\begin{pmatrix}1&&\\&1&\\&1&1\end{pmatrix}$ $E_6=\begin{pmatrix}1&&-1\\&1&\\&&1\end{pmatrix}$ $\underline{A=E_1^{-1}\cdots E_6^{-1}}$

例3 对 $T,U: V\to W$, a) $\mathrm{Range}(T+U)\subseteq R(T)+R(U)$ b) $\mathrm{rank}(T+U)\leq \mathrm{rank}\,T+\mathrm{rank}\,U$

□ a) $y=T(x)+U(x)$ for some x b) $\underline{\mathrm{rank}\,T+U \overset{3.\,T_3}{\leq} \dim(R(T)+R(U))} \overset{P12.\,T_{20}}{=} \dim \underbrace{R(T)+\dim R(U)-\dim(R(T)\cap R(U))}_{\leq \mathrm{rank}\,T+\mathrm{rank}\,U}$

NOTE 这说明 $\mathrm{rank}(A+B)=\mathrm{rank}(L_{A+B})\overset{\boxed{四}.P_2\,iii)}{=}\mathrm{rank}(L_A+L_B)\leq \mathrm{rank}\,A+\mathrm{rank}\,B$

例4. $A_{m\times n}$, $B_{n\times p}$, $\mathrm{rank}\,A=m$, $\mathrm{rank}\,B=n$, 求 $\mathrm{rank}(AB)$

□ $L_A(\mathbb{F}^n)=\mathbb{F}^m$, $L_B(\mathbb{F}^p)=\mathbb{F}^n$, 则 $L_{AB}(\mathbb{F}^p)=L_AL_B(\mathbb{F}^p)=L_A(\mathbb{F}^n)=\mathbb{F}^m$, ∴ rank = m

例5. 若 A m×n rank m, ∃ $B_{n\times m}$ st. $AB=I_m$; 相应地, 若 $B_{n\times m}$ rank m, ∃ $A_{m\times n}$ st. $AB=I_m$

□ i) 即 $R(L_A)=\mathbb{F}^m$, onto. 对 $\mathbb{F}^m$ basis $\{e_1\cdots e_m\}$, 对 e_i, ∃ $L_A(\underset{=}{\overset{\in\mathbb{F}^n}{v_i}})=e_i$, 令 $B=(v_1\cdots v_m)$

$AB=(Av_1\cdots Av_m)\overset{\boxed{四}3}{=}(L_A(v_1)\cdots L_A(v_m))=(e_1\cdots e_m)=I_m$

ii) 考虑 $\underline{\mathrm{rank}(B^T)=m}$ (超 P i)) ∴ B^T 为 m×n, rank = m, 由 i), ∃ C st. $B^TC=I_m=(C^TB)^T$ ∴ 令 $C^T=A$

例6. i) A 3×1, B 1×3, 则 AB at most rank 1
ii) C 3×3 rank 1, 则 ∃ A 3×1, B 1×3, st. C = AB

□ i) 令 $\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot(b_1\,b_2\,b_3)=\begin{pmatrix}a_1b_1&a_1b_2&a_1b_3\\a_2b_1&a_2b_2&a_2b_3\\a_3b_1&a_3b_2&a_3b_3\end{pmatrix}$, 可知各 col 比例关系 (eg. 2r 为 1r × $\frac{a_2}{a_1}$) ∴ rank ≤ 1

ii) 令 C 1 row L.i.d. 令 $C=\begin{pmatrix}-c_1-\\-ac_1-\\-bc_1-\end{pmatrix}$, ∴ $A=\begin{pmatrix}1\\a\\b\end{pmatrix}$, $B=c_1$

例7. 对 $A=\begin{pmatrix}1&0&-1&2&6\\-1&1&3&-1&0\\-2&1&4&-1&3\\3&-1&-5&1&-6\end{pmatrix}$, i) M 5×5 rank 2, AM = 0 , ii) B 5×5, AB = 0, 证 B rank ≤ 2

□ i) 令 $M=(m_1\,m_2\,0\,0\,0)$, AM=0 即 $Am_1=0\in\mathbb{F}^4$, $Am_2=0\in\mathbb{F}^4$. 解方程组: 求 $(A|0)$ 来求 $Ax=0$

null space 为 $\left\{\begin{pmatrix}x_3+3x_5\\-2x_3+x_5\\x_3\\-2x_5\\x_5\end{pmatrix}\right\}$, basis 为 $\left\{\begin{pmatrix}1\\-2\\1\\0\\0\end{pmatrix},\begin{pmatrix}3\\1\\0\\-2\\1\end{pmatrix}\right\}$ ∴ 可令 $M=\begin{pmatrix}1&3&1&1&1\\-2&1&1&1&1\\1&0&0&0&0\\0&-2&1&1&1\\0&1&1&1&1\end{pmatrix}$

ii) 令 $B=(b_1\,b_2\,b_3\,b_4\,b_5)$ 即各 b_i 在 $Ax=0$ 的解空间, 但对 L_A: $\underline{\overset{5}{\dim(\mathbb{F}^5)}}=\overset{3}{\mathrm{rank}(L_A)}+\mathrm{nullity}(L_A)$ ∴ $\mathrm{nullity}(L_A)=2$, 而 $b_i\in N(L_A)$ ∴ rank ≤ 2

证1. $\mathrm{rk}\,A^TA=\mathrm{rk}\,AA^T=\mathrm{rk}\,A=\mathrm{rk}\,A^T$ (Gram matrix rk eq)

□ 由 $\boxed{10}.5$ & $\boxed{四}.3$

四.3

研究对 matrix representation 的 L.T. 的性质. 由 $\text{rank}(A)=\text{rank}(L_A)$

1. T_{1a} A m×n, 有 P m×m, Q n×n 可逆, i) $\text{rank}(AQ)=\text{rank}\,A$ ii) $\text{rank}(PA)=\text{rank}\,A$ iii) $\text{rank}(PAQ)=\text{rank}\,A$

四.P_2 ii) Def of range, L_Q onto, $Q=[L_Q]_\beta$ 可逆, L_Q 可逆

□ i) 由 $\text{Rnge}(L_{AQ})=\underline{\text{Range}(L_AL_Q)}=L_AL_Q(F^n)=L_A(L_Q(F^n))=L_A(F^n)=R(L_A)$ (四.1)

∴ $\text{rank}(AQ)=\dim(\text{Rnge}(L_{AQ}))=\dim(\text{Rnge}(L_A))=\text{rank}\,A$ 秩定def

ii) $R(L_PL_A)=L_PL_A(F^n)=L_P(\text{Range}(L_A))$

∴ $\text{rank}(PA)=\dim L_P(R(L_A))=\dim(R(L_A))=\text{rank}(A$

对 这即说 elementary row/col operation 不改变 rank (四.T_2, E 可逆) (L_P isomorphism, 四.1, T 为 L_P)

T_{1b} rank of matrix 同 max # of L.Ind. col (四.3 的内容) 四.3.ii)

□ $\text{rank}\,A=\overset{\text{秩定def}}{\text{rank}\,L_A}=\dim(R(L_A))=\dim(L_A(\beta))\overset{10.T_2}{=}\dim(\text{span}\{A(e_i)\}_i)=\dim(\text{span}\{A_{:,i}\}_i)$

T_{1c} let A m×n. A rank $=r\Rightarrow\exists$ 可逆 B, C st. $\begin{pmatrix} I_r & O\\ O & O\end{pmatrix}=D=BAC$

想象成 composite / $AB\ B^{-1}A^{-1}$ $(AB)^{-1}$..

□ 经过初等变换 st. $A\rightarrow\begin{pmatrix} 1 & & 0\\ & \ddots & \\ 0 & & 0\end{pmatrix}$ 形式, 即 $D=\underbrace{E_p\cdots E_1}_{B}A\underbrace{F_1\cdots F_q}_{C}$, 而可逆乘积仍可逆

且有 $r\leq 1$ (operation 不改变 rank, & T_{1b}) D 的性质 st. rank 为 r

P i) $\text{rank}\,A^T=\text{rank}\,A$ 即使由四.3, Nullity(A) ≠ Nullity(A^T) 因 A 可能不同 # col 四.5.3, BC 未可逆

□ 由 T_{1c}, 有 $D=BAC$, ∴ $D^T=(BAC)^T=C^TA^TB^T$, 因 C^T, B^T 可逆, 由 T_{1a} iii) $\text{rank}(C^TA^TB^T)=\text{rank}\,A^T=\text{rank}\,D^T\overset{T_{1c}}{=}r=\text{rank}\,A$

ii) (T_{1b} 拓展) rank of matrix 同 max # of L.Ind. row

□ max # of L.Ind row 同 max # of L.Ind col of A^T, 而 $\text{rank}\,A^T\overset{P_i)}{=}\text{rank}\,A$

Note T_{1b} 和本性质说明: row/col generate 的 subspace dim 为 rank (col space)

iii) 可逆矩阵为 elementary matrix 乘积 四 T_2 inverse elementary 仍 elementary

□ A n×n 可逆, 则 rank = n $\text{Null}(L_A)=\{0\}$ (1-1) ∴ $\text{rank}(L_A)=n=\text{rank}\,A$ (秩定def)

∴ 由 T_{1c}, $D=I_n$, 且有 $D=I_n=BAC=\underbrace{E_p\cdots E_1}_{B}A\underbrace{F_1\cdots F_q}_{C}$, ∴ $A=\underbrace{E_1^{-1}\cdots E_p^{-1}F_q^{-1}\cdots F_1^{-1}}$

iv) $\text{rank}\,A=\text{rank}\,kA$, k nonzero scalar. □ WTS $\text{Range}(L_A)=R(L_{kA})$, 而 $R(L_{kA})=L_{kA}(F^n)=kL_A(F^n)=L_A(F^n)=R(L_A)$

OR 由 T_{1a} 易 四.P_2 iii) 对 $L_A(x)$ 有 $kL_A(x/k)$ (L_A L.T.

T_2 T: V→W, U: W→Z; A B 为矩阵, 证: a) $\text{rank}(UT)\leq\text{rank}(U)$ b) $\text{rank}(UT)\leq\text{rank}\,T$ c) $\text{rank}(AB)\leq\text{rank}\,A$ d) $\text{rank}\,AB\leq\text{rank}\,B$

□ a) $R(UT)=UT(V)=U(\underbrace{R(T)}_{R(T)\subseteq W})\subseteq U(W)=R(U)$, $\text{rank}\,UT\leq\text{rank}\,U$ 3.T_3

c) $\text{rank}(L_{AB})=\text{rank}(L_A\circ L_B)\leq\text{rank}(L_A)$ (由 a))

d) $\text{rank}\,AB=\text{rank}\,AB^T\overset{P_i)}{=}\text{rank}\,B^TA^T\overset{由\,c)}{\leq}\text{rank}\,B^T=\text{rank}\,B$

b) 令 $A=[U]_\beta^\gamma$, $B=[T]_\alpha^\beta$, 则 $AB\overset{四.1}{=}[UT]_\alpha^\gamma$, ∴ $\text{rank}\,AB=\text{rank}\,UT\leq\text{rank}\,B=\text{rank}\,T$ d)

$\text{rank}\,T=\text{rank}(L_{[T]_\alpha^\beta})=\text{rank}(L_B)=\text{rank}\,B$ 四.2 秩定def

即 $\text{rank}(AB)\leq\min\{rk(A), rk(B)\}$

T_3 (Frobenius) $rk\,AB+rk\,BC\leq rk\,B+rk\,ABC$

□ $rk\,AB=rk\,B-\dim(R(B)\cap N(A))$ (四.导) 又, 令 $B\leftarrow BC$, $rk\,ABC=rk\,BC-\dim(R(BC)\cap N(A))$

∴ $rk\,AB+rk\,BC=rk\,B+rk\,ABC+\underbrace{\dim R(BC)\cap N(A)-\dim R(B)\cap N(A)}$

≤ 0 因 $R(BC)\subseteq R(B)$ 左≤右

NOTE 当 $B\leftarrow I_n$, $C\leftarrow B$ 为 (Sylvester) $rk\,AB\geq rk\,A+rk\,B-n$

Date. / 即有无eigval的L.T.

例1 a) T: rotate $\mathbb{R}^2$ by $\pi/2$ b) 令 $C^\infty(\mathbb{R})$ = {set of func: $\mathbb{R}\to\mathbb{R}$ | f有任意阶导数} $\subseteq \mathcal{F}(\mathbb{R},\mathbb{R})$

T: $C^\infty(\mathbb{R})\to C^\infty(\mathbb{R})$, $f\to f'$ 找出 eigenvector

□ a) 几何角度，无T(v)为v的伸展(kv) ∴ 无 eigenvector, not $\overset{\text{无此}\beta}{\underline{\text{diagonalizable}}}$

b) 若f eigenvec. 有 $T(f)=f'=\lambda f$ 1-order DE: $f(t)=ce^{\lambda t}$. ∴ f在 $c\neq 0$ 时为 eigenvector 对应λ为∀ real ($\lambda=0$, f为 zero func)

例2: V Vec. Space, k scalar, kI_V diagonalizable?

□ note 对 $\underline{\underline{\forall}}$ basis β, $[kI_V]_\beta = kI$, 对角矩阵 ∴ 无论何basis都是. (由f(λ), 所有一 eigenval = k)

若A similar to $\underset{\text{scalar matrix}}{\underline{\underline{kI}}}$, 则 $A=kI$; 且 diagonalizable 只有一 eigenval 的为 scalar matrix

□ $A=P^{-1}kIP = kP^{-1}IP = kI$; 令 $D=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$ 只有一 eigenval, $\lambda_1=\dots=\lambda_n=\lambda$

由上证 $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ 无法 diagonalizable □ $f(\lambda)=(1-\lambda)(1-\lambda)$, 一 eigenval, 若可 diagonalizable, 为scalar matrix, False

例3. 证无 basis in $\mathbb{R}^2$ st. $[T]_\beta = A = \begin{pmatrix}1 & 2\\ 2 & 6\end{pmatrix}$, $[T]_\gamma = B = \begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}$

□ 令 $B = \underset{\gamma\to\beta}{Q^{-1}} A \underset{\gamma\to\beta}{Q}$, $\det(C-\lambda I) = \det(Q^{-1}AQ - \lambda Q^{-1}IQ) = \det(Q^{-1}(A-\lambda I)Q)$

$= \det Q^{-1} \det(A-\lambda I) \det Q = \det A\lambda I$, 即 characteristic poly 相同, 但 $f_A(t)=\lambda^2-7\lambda+2 \neq f_B(t)=\lambda^2-5\lambda+5$

例4. 若 $T: M_2(\mathbb{R})\to M_2(\mathbb{R})$, $A\to A^T$. 求T 特征值

□ 令 $T(A)=\lambda A$, $A^T=\lambda A$, 而 $T(A^T)=T(\lambda A)=A=\lambda^2 A$ ∴ $\lambda=\pm 1$.

$\lambda=1$, eigspa = {symmetric} $\lambda=-1$ eigspa = {skew-symmetric}, dim 见 P_{12} 例1

例5 证 real symmetric 的 eigenval 也 real

□ 对 $Ax=\lambda x$, $\langle Ax, x\rangle = \langle\lambda x, x\rangle = \lambda\langle x, x\rangle$

也 $= \bar{\lambda}\langle x, x\rangle$ (P_{49} P_3 ii) ∴ $(\lambda-\bar{\lambda})\langle x, x\rangle = 0$ (只有x=0 才为0, 且 eigenvec ≠ 0) ∴ $\lambda=\bar{\lambda}$

例6, real symmetric 的 distinct eigenvec 且 mutually orthogonal

□ 对 $\lambda_1\neq\lambda_2$, $Av_1=\lambda_1 v_1$, $\langle Av_1, v_2\rangle = \langle\lambda_1 v_1, v_2\rangle = \lambda_1\langle v_1 v_2\rangle$

$\lambda_1\neq\lambda_2$ ↑ $Av_2=\lambda_2 v_2$

也 $= \underset{v_1^T A^T v_2)}{(Av_1)^T v_2} = \langle v_1, \underset{\text{symmetric} = A}{A^T} v_2\rangle = \langle v_1, \lambda_2 v_2\rangle$

∴ $\underline{\lambda_1\langle v_1, v_2\rangle = \lambda_2\langle v_1 v_2\rangle}$ ∴ $\langle v_1, v_2\rangle = 0$

证1. $tr(A)=\lambda_1+\dots+\lambda_n$ 法1 ∀ matrix similar to Jordan Form, similar matrix share eigvalue,

∴ $tr(A)=tr(P^{-1}JP)=tr(PP^{-1}J)=tr(J)=\sum\lambda_i(J)=\sum\lambda_i(A)$ 法2 由右 J_{1n} k_{n-1} 由$\lambda_1+\dots+\lambda_n$组成

证2 skew-symmetric $A^T=-A$ 为 eval 0 or purely imaginary

□ $x\in\mathbb{C}^n$ $x^T A x \begin{cases} = \lambda x^T x = \lambda\|x\|^2 \\ = (A^T \bar{x})^T x = \bar{\lambda}\|\bar{x}\|^2 \end{cases}$ ⇒ $\lambda=-\bar{\lambda}$

conjugate $\bar{\lambda}\ \bar{x}$

diagonalization & eigen 3 ~~example 1.1~~

即对A，先change basis为β, st. D为diagonal，再变换回来
转 def

回中，令 n阶方阵A为$[T]_{\alpha}$ standard basis，若T diagonalizable，有$[T]_{\beta}$ diagonal
即 $A = \underset{\beta \to \alpha}{Q} D \underset{\beta \to \alpha}{Q^{-1}}$ (4 T_{5a})

1. $T: V \to V$ diagonalizable iff ∃ ordered basis β st. $[T]_{\beta}$ diagonal
令 方阵 A diagonalizable iff L_A diagonalizable 即 similar to diagonal
2.2. 的def 对应

Note, 若 $\underset{\in \beta}{D} = [T]_{\beta}$ diagonal, 对 $\forall v_j \in \beta$, $T(v_j) = \sum_i D_{ij} v_i = D_{jj} v_j = \lambda_j v_j$
而若 $T(v_j) = \lambda_j v_j$，则 $[T]_{\beta} = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ $[[T(v_1)]_{\beta} \; [T(v_2)]_{\beta} \dots]$ $\Big\}$ T diagonalizable ⟺ $T(v_j) = \lambda_j v_j$

称 $v \in V$ eigenvector of T ⟺ $\exists \lambda \in \mathbb{F}$, $T(v) = \underset{\text{eigenvalue}}{\lambda} v$

对应地, $A_{n \times n}$, $v \in \mathbb{F}^n$ eigenvector of A ⟺ v eigenvector of L_A

∴ 以 eigenval/eigenvec 表述, diagonalizable 指 T 的basis $\beta = \{v_1 \dots v_n\}$ 为set of eigenvector, 且因β basis, v_i distinct nonzero

因β basis, 说明$A_{n,n}$ ($L_A: \mathbb{F}^n \to \mathbb{F}^n$) 不会超过n个 eigenval

记为$f(\lambda)$, characteristic polynomial

2. λ eigenval ⟺ $\det(A - \lambda I_n) = 0$ 回.2 note: $A-\lambda I$ nullity > 0, $rk(A-\lambda I) < n$
回.4.T_1 P_7.T_1

☐ ⟺ ∃v 使 $Av = \lambda v$ ⟺ $(A - \lambda I_n)(v) = 0$ ⟺ $A - \lambda I_n$ not invertible ⟸ $\det(A - \lambda I_n) = 0$

对应的, characteristic polynomial of T 为 $f(\lambda) = \det([T]_{\beta} - tI_n)$
且 ~~$f(\lambda)$ indep. of chosen basis~~
因 回 3 即使不同basis, 两 matrix representation similar, ∴ $f(\lambda)$ 相同

T_{1a} characteristic polynomial 为 degree n, 且leading coeff 为 $(-1)^n$

☐WTS $f(\lambda) = (-1)^n \lambda^n + k_{n-1} \lambda^{n-1} + \dots + k_1 \lambda + k_0$. In fact, by induction $f(\lambda) = (A_{11} - t) \dots (A_{nn} - t) + q(\lambda)$, 同

$q(\lambda)$ degree ≤ n-2: 令 对n-1成立, 对n, $\begin{vmatrix} A_{11}-\lambda & & \\ & \ddots & \\ & & A_{nn}-\lambda \end{vmatrix}$ $\overset{\text{最末行 expand}}{=}$ $(A_{nn} - \lambda) \underbrace{\begin{vmatrix} A_{11}-\lambda & & \\ & \ddots & \\ & & A_{n-1,n-1}-\lambda \end{vmatrix}}_{\text{满足归纳条件}} + \sum_{i=1}^{n-1} (-1)^{n+i} A_{ni} \cdot \det \tilde{A}_{ni}$
degree ≤ n-3
去掉 n row i col
即无 $A_{nn} - \lambda$, $A_{ii} - \lambda$, ∴ degree at most n-2

$= (A_{nn} - \lambda)(A_{11} - \lambda) \dots (A_{n-1,n-1} - \lambda) + (A_{nn} - \lambda) q(\lambda) + \underset{\text{degree} \le n-2}{q'(\lambda)}$

∴ 展开 $(A_{11} - \lambda) \dots (A_{nn} - \lambda)$, λ coeff 为 $(-1)^n$
$k_{n-1} = (-1)^{n-1} tr(A)$
NOTE, $f(0) = k_0 = \det(A - 0 \cdot I_n) = \det(A)$. 见 P_{7} T_{4}

T_{1b} $A_{n,n}$ 最多n distinct eigenvalues ☐ 由T_{1a}, eigenval 为 $f(\lambda)$ 的zero. ∴ 最多n个不同个解

T_{1c} 对 $T: V \to V$, λ为 eigenvalue of T. 则 $v \in V$ eigenvector ⟺ $v \ne 0 \wedge v \in Null(T - \lambda I)$

☐ ⟹ $v \ne 0$, 且有 $T(v) = \lambda v$. ∴ $T(v) - \lambda v = (T - \lambda I)(v) = 0$ ⟸ 即 $(T - \lambda I)(v) = 0$, $T(v) = \lambda v$ 即 E_{λ} (回.4)

3. T_{2a} let $\lambda_1 \dots \lambda_k$ distinct eigenval of T, $\{v_1 \dots v_k\}$ L.ind

☐ induction on k, k-1 T, 则对k, let $\sum_{i=1}^{k} a_i v_i \overset{①}{=} 0$, 两边 apply func: $T - \lambda_k I$: $a_1[T(v_1) - \lambda_k I(v_1)] + \dots = 0$

$a_1 (\lambda_1 - \lambda_k) v_1 + \dots + a_k 0 = 0$ ∵ $\{v_1 \dots v_{k-1}\}$ L.ind. ∴ $a_1(\lambda_1 - \lambda_k) = \dots = a_{k-1}(\lambda_1 - \lambda_{k-1}) = 0$ 而 λ_i distinct
∴ $\lambda_i - \lambda_k \ne 0$ ∴ $a_1 = \dots = a_{k-1} = 0$

则①: $a_k v_k = 0$ 但 v_k eigenvector, ∵ $v_k \ne 0$ ∴ $a_k = 0$

T_{2b} T有n distinct eigenval ⟹ T diagonalizable ☐ $\underset{T_{2a}}{\{v_1 \dots v_n\}}$ L.ind. ∴ 由1.知, 有basis of eigenvec. ∴ T diagonalizable

Note T_{2b} 逆非true. I 的eigenval 只有1, 但 diagonalizable. 例见 回.例3.1

4. diagonalization 的~~好处~~ 是 决定 matrix power, 若 $\overset{\text{核心}}{A = PDP^{-1}}$, $A^k = PD^kP^{-1}$, 如 回14.例3

$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ $P = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ 因P 这 eigen decomposition ⟺ $\{v_i\}$ 为 basis

且 $\det A = \det P \det D \det P^{-1} = \det D = \prod \lambda_i$ 同 result as P_7.T_4

diagonal 2 & triangular

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1. diagonal matrix $\begin{pmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_n \end{pmatrix}$ 记为 diag(v),

易知 对 $x \in \mathbb{F}^n$, $\text{diag}(v) \cdot x = \sum x_i v_i = v \odot x$

且 $\text{diag}(v)^{-1} = \text{diag}([\frac{1}{v_1} \cdots \frac{1}{v_n}]^T)$

triangular matrix $\begin{pmatrix} a & & \\ & b & \\ 0 & & \ddots \end{pmatrix}$ 易用cofactor expansion 证 det $= \Pi$ diag$_i$

strict triangular $\begin{pmatrix} 0 & & \\ & 0 \ddots & \\ 0 & & 0 \end{pmatrix}$ main diagonal $= 0$, det $= 0$

tri $\underset{\times}{+}$ tri 仍为 tri

2. 性(a) square A nilpotent iff $\exists\, m \geq 1$ s.t. $A^m = 0$ □ characteristic $f(\lambda) = (-1)^n \lambda^n$

由 Cayley-Hamilton $f(A) = (-1)^n A^n = 0$

strict triangular nilpotent : $A^n = 0$

(b) triangular A inverse 仍为 triangular (if 可逆) □ 法1 $A^{-1} = \frac{1}{\det A} \text{adj} A$ 见 1.3 若 A $\overset{\text{upper}}{\underset{\text{lower}}{}}$ triangular, $C_{ij} = 0$ $\overset{i<j}{\underset{i>j}{}}$

note C_{ij} 顺序 → ∴ $\text{adj}(A)$ $\overset{\text{upper}}{\underset{\text{lower}}{}}$

法3 对上△ A 拆为 $\underset{z}{\overset{\text{diag}}{D}}(I+\overset{\text{strict 上△}}{B})$, $A^{-1} = (I+B)^{-1} D^{-1}$, 而 $\overset{\text{upper. P10.3}}{(I+B)^{-1}} = I - B + B^2 - \cdots + (-1)^{n-1} B^{n-1}$ (之后为0 因 $B^n = 0$)

若 A lower triangular

法2 令 $A^{-1} = [a_1' \cdots a_n']$ $AA^{-1} = [Aa_1' \cdots Aa_n'] = I$ ∴ $Aa_k' = e_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow k$ thus a_k' 在 k row 上 … 为 0

∴ A^{-1} lower triangular

Note 可逆 $\Leftrightarrow$ det $\neq 0$ $\Leftrightarrow$ diag$_i \neq 0$

$\|x\|\le 1$ stretch

measure 可以将 unit ball vector transform 成 ellipsoid 的 size

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因 $\|A\|_{op} \ge \forall x\neq 0 \ \frac{\|Ax\|}{\|x\|}$ ∴ $\|A\|_{op}$ 也为 induced norm 之4。 ab 2 norm

1. (operator norm) $\|A\|_{op} = \sup_{x\neq 0}\frac{\|Ax\|}{\|x\|} = \inf\{c\ge 0 \mid \|Ax\| \le c\|x\|, \forall x\} = \|A\|_{ab} = \sup_{\|x\|_b\le 1}\|Ax\|_a$

证明为 norm □ 有 property i) $\forall A, x$ 有 $\|Ax\|_a \ge 0$, $\forall A \ \sup_x \|Ax\|_a \ge 0$ ∴ $\forall A \ \sup_{\|x\|_b\le 1}\|Ax\|_a\ge 0$ 若A≠0, 己∃ [illegible] $Ax\neq 0$

当 $A=0$, $\|0\|_b = \sup_{\|x\|_b\le 1}\|0x\|_a = 0$, 而若 $\|A\|_{ab}=0$, def $\sup_{\|x\|_b\le 1}\|Ax\|_a = 0$, ∴ $\forall \|x\|_b\le 1$ 有 $\|Ax\|_a=0 \xRightarrow{\|\cdot\|_a \text{的 i)}} Ax=0 \Rightarrow A=0$

ii) $\sup\|\alpha Ax\|_b \overset{\|\cdot\|_b \text{的 ii)}}{=} \sup|\alpha|\,\|Ax\|_b = |\alpha| \sup\|Ax\|_b$

iii) 从RHS $\|A\|_{ab}+\|B\|_{ab} = \sup\|Ax\|_a + \sup\|By\|_a \ge \|Ax\|_a+\|By\|_a$, $\forall \|x\|_b, \|y\|_b \le 1$. 因 x y 任意, 取 $u=x=y$ 以 $\|u\|_b\le 1$, 有上式 $=\|Au\|_a+\|Bu\|_a$

$\overset{\|\cdot\|_a \text{的 iii)}}{\ge} \|Au+Bu\|_a$ ∴ by def $\|A\|_{ab}+\|B\|_{ab} \ge \boxed{\sup_{\|u\|_b\le 1}\|Au+Bu\|_a} \overset{\text{def of sup}}{=} \|A+B\|_{ab}$

2. T i) $\forall \|\cdot\|_a$ cont. on V.S. V under topology induced by $\|\cdot\|_1$ (unif. cont.)

□ WTS $\forall\varepsilon>0 \ \exists\delta>0$ s.t. $\|x-x'\|_1<\delta \Rightarrow \big|\|x\|_a-\|x'\|_a\big|<\varepsilon$ of 本P2 v)

由有 property iii) on $\|\cdot\|_a$ $\left|\begin{cases}\|x\|_a-\|x'\|_a = \|x'+(x-x')\|_a-\|x'\|_a \le \|x-x'\|_a \\ \|x'\|_a-\|x\|_a = \|x-(x-x')\|_a-\|x\|_a \le \|x-x'\|_a\end{cases}\right.$ ∴ RHS $\le \|x-x'\|_a$

因 x under V 有 basis $\{e_i\}^n$ s.t. $x=\sum^n\alpha_ie_i$, $x'=\sum^n\alpha_i'e_i$, $\|x-x'\|_a \le \sum^n|\alpha_i-\alpha_i'|\cdot\|e_i\|_a \le \|x-x'\|_1\cdot\boxed{\overset{\max\|e_i\|_a}{M}}$

∴ 若令 $\delta=\frac{\varepsilon}{M}$ $\|x-x'\|_1<\delta$ 有 $\big|\|x\|_a-\|x'\|_a\big| \le \|x-x'\|_a \le \|x-x'\|_1\cdot M<\varepsilon$

ii) (norm equivalent) $\forall 2 \ \|\cdot\|_a, \|\cdot\|_b$ on V.S. V equiv, 即 $\exists$ real $0<C_1\le C_2$ s.t. $\forall x\in V$, $C_1\|x\|_b\le\|x\|_a\le C_2\|x\|_b$

Step 1) (norm transitivity) 若 $\|\cdot\|_a, \|\cdot\|_b$ equiv $\|\cdot\|_1$, 则 $\|\cdot\|_a$ equiv $\|\cdot\|_b$

□ 有 $C_1\|x\|_1\le\|x\|_a\le C_2\|x\|_1$ ∴ $\frac{C_1'}{C_2}\|x\|_a\le\|x\|_b\le\frac{C_2'}{C_1}\|x\|_a$ ∴ equiv.
$C_1'\|x\|_1\le\|x\|_b\le C_2'\|x\|_1$

Step 2) 只需证 $C_1\|x\|_1\le\|x\|_a\le C_2\|x\|_1$, 若 $x=0$ trivial, 若 $x\neq 0$, 取 $u=\frac{x}{\|x\|_1}$ 则WTS $C_1\le\|u\|_a\le C_2$

Note $\|u\|_1=1$, 由 Extreme Value Thm, cont. $\overset{\text{T i)}}{\|\cdot\|_a}$ on compact set $\overset{\text{close \& bounded}}{\{u \mid \|u\|_1=1\}}$ 必有 max min in set

∴ 令 $C_1=\min_{\|u\|_1=1}\|u\|_a$ 因 $u\neq 0$, $C_2\ge C_1>0$, 且 $C_1\le\|u\|_a\le C_2$
$C_2=\max_{\|u\|_1=1}\|u\|_a$

3. 证 $\|A\|$ induced norm 为 matrix norm

□ i) 对 $A\neq 0$ 有 $\|x\|=1$, s.t. $Ax\neq 0$, ∴ $\underset{\text{vec norm def}}{\|Ax\|\neq 0}$, 又若 $A=0$ 无论x如何 $Ax=0$, ∴ $\|Ax\|=0$

ii) $\|cA\| = \max\|cAx\| = \max|c|\,\|Ax\| = |c|\max\|Ax\|$ 对 induced) $\forall y\neq 0$ 有 $\vec{x}=\frac{y}{\|y\|}$ 满足 $\|x\|=1$

iii) 可找到 $\|x_0\|=1$ & $\|A+B\|=\|(A+B)x_0\|$, ∴ $\|Ax_0+Bx_0\| \le \|Ax_0\|+\|Bx_0\|$ (iii) of vec. norm) ∴ $\|Ay\| = \|A\|y\|x\| = \|y\|\cdot\|Ax\|$ $\le \|y\|\cdot\|A\|$

(由 induced) of matrix norm) $\le \|A\|\|x_0\|+\|B\|\|x_0\|$ iv) 可 find $\|x_0\|=1$, 有 $\|ABx_0\|=\|AB\|$

∴ $\|ABx_0\| \overset{\text{induced) of matrix norm}}{\le} \|A\|\cdot\|Bx_0\| \le \|A\|\,\|B\|\,\|x_0\|$

recall spectral 2 P25 定理 $A^TA = UDU^T$ spectral decomp $[A^TAx]_x = [A^TA]_x[x]_x \doteq Q^{-1}_{x\to e}[UDU^T]_e$ ~~$[x]_x$~~ $= U^{-1}UDU^TU[x]_x = D[x]_x = \sum c_i\lambda_ix_i$ note $U=Q_{x\to e}$

4. matrix norm induced by vec norm $\|x\|_2=\sqrt{\langle x,x\rangle}$, 有 $\|A\|=\sqrt{\lambda_1}=\sigma_1$, λ_1 为 A^TA 的 largest eval → $\sigma_1(A)$ max singular v

□ $\|Ax\|^2 = \langle x, A^TAx\rangle$ 因 A^TA psd Optim P6. P1 b), 有 $\lambda_1\ge\dots\ge\lambda_n\ge 0$ & orthonormal evec basis $\{x_i\}$ 1's o. 1

取 $\|x\|=1$, $= \langle c_1x_1+\dots+c_nx_n, c_1\lambda_1x_1+\dots+c_n\lambda_nx_n\rangle$ 令 x 由 $\{x_i\}$ 表示, 因 $\|x\|=1$ 有 $c_1^2+\dots+c_n^2=1$

$\{x_i\}$ orthonormal $= \lambda_1c_1^2+\dots+\lambda_nc_n^2 \le \lambda_1(c_1^2+\dots+c_n^2)=\lambda_1$, 等号当 $x\leftarrow x_1$ 取到 $\langle x_1, A^TAx_1\rangle = \langle x_1, \lambda_1x_1\rangle = \lambda_1$

∴ $\|A\| = \max\|Ax\| = \sqrt{\lambda_1}$

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5. (Rayleigh) 对 P pd 有 $\lambda_{min}\|x\|^2 \le x^TPx \le \lambda_{max}\|x\|^2$ 见 P63 T1

Inner product & norm

norm 与metric space (问题) $d(x,y)=\|x-y\|$

0. 数学上 def norm $\|x\|$, $\|\cdot\|: \mathbb{R}^n \to \mathbb{R}$
i) $\|x\|>0$ if $x\neq\vec{0}$, $\|x\|=0 \Leftrightarrow x=\vec{0}$
ii) $\|\alpha x\| = |\alpha|\,\|x\|$
iii) (triangle) $\|x+y\| \leq \|x\|+\|y\|$

L_1 $\|x\|_{\ell_1} = \sum_i |x_i|$
ℓ_2 $\sqrt{\sum_i |x_i|^2}$
ℓ_p $\sqrt[p]{\sum_i |x_i|^p}$
ℓ_∞ $\max_i |x_i|$

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1 $\langle\cdot,\cdot\rangle: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ 为 inner product iff
a. positivity $\langle x,x\rangle \geq 0$ ($x=0$ 等号成立)
b. symmetry $\langle x,y\rangle = \langle y,x\rangle$
c. additivity $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$
d. homogeneity $\langle rx,y\rangle = r\langle x,y\rangle$, $r\in\mathbb{R}$

P_1 i) $\langle x,y+z\rangle = \langle y+z,x\rangle = \langle y,x\rangle + \langle z,x\rangle = \langle x,y\rangle + \langle x,z\rangle$
ii) $\langle x,ry\rangle = \langle ry,x\rangle = r\langle y,x\rangle = r\langle x,y\rangle$

2. def Euclidean inner product $\langle x,y\rangle = \sum^n x_i y_i = x^T y$
且 norm $\|x\| = \sqrt{\langle x,x\rangle} = \sqrt{x^T x}$

P_2 i) (Cauchy-Schwarz) $|\langle x,y\rangle| \leq \|x\|\,\|y\|$, $x=\alpha y$ 等号成立, $\alpha\in\mathbb{R}$
即 $(\sum x_i y_i)^2 \leq (\sum x_i^2)(\sum y_i^2)$

□ 若 $\|x\|=\|y\|=1$, $\underbrace{0 \leq \|x-y\|^2}_{1.a} = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2 = 2-2\langle x,y\rangle$, $\therefore \langle x,y\rangle \leq 1$ 等号 $x=y$ 1.d

当 x,y 有 $=0$, 显然
当 $x,y\neq 0$. 令 $x' \leftarrow x/\|x\|$, $y' \leftarrow y/\|y\|$, now $\|x'\|=\|y'\|=1$ $\therefore \langle x',y'\rangle = \frac{\langle x,y\rangle}{\|x\|\cdot\|y\|} \leq 1$

且令 $x' \leftarrow -x$, $\langle x',y\rangle = \underbrace{-\langle x,y\rangle \leq \|x\|\cdot\|y\|}_{上行末}$ $\therefore |\langle x,y\rangle| \leq \|x\|\cdot\|y\|$ 等号 $\frac{x}{\|x\|} = \frac{\pm y}{\|y\|}$

证明此 $\|\cdot\|$ 确为 norm:

ii) $\|x\| \geq 0$, 等号 $x=0$ 1.a
iii) (homogeneity) 对 $r\in\mathbb{R}$, $\|rx\| = |r|\,\|x\|$ □ $\sqrt{\langle rx,rx\rangle} = \sqrt{r^2\langle x,x\rangle}$ 1d
iv) (三角不等式) $\|x+y\| \leq \|x\| + \|y\|$
□ $\|x+y\|^2 = \|x\|^2 + 2\overset{\leq \|x\|\|y\|\ i)}{\langle x,y\rangle} + \|y\|^2 \leq (\|x\|+\|y\|)^2$
v) $\big|\,\|x\| - \|y\|\,\big| \leq \|x-y\|$
□ $\|x\| = \|x-y+y\| \overset{iv)}{\leq} \|x-y\| + \|y\|$, $\therefore \|x\| - \|y\| \leq \|x-y\|$ 同理 $\|y\| - \|x\| \leq \|y-x\| \overset{iii)}{=} \|x-y\|$

Note $\|\cdot\|$ 为 unif. cont: $\forall \varepsilon>0$, 令 $\delta=\varepsilon$, 当 $\|x-y\| < \delta = \varepsilon$, 有 $\big|\,\|x\| - \|y\|\,\big| \leq \|x-y\| < \varepsilon$ 见定理 i)

3. 对 $\mathbb{C}^n$, inner product $\langle x,y\rangle := \sum^n x_i \overline{y_i}$

P_3 i) $\langle x,y\rangle = \overline{\langle y,x\rangle}$ □ 而右 $= \overline{\sum y_i \overline{x_i}} = \sum \overline{y_i}\,\overline{\overline{x_i}} = \sum \overline{y_i}\,x_i$ conj 性质
ii) $\langle x, r_1 y + r_2 z\rangle = \overline{r_1}\langle x,y\rangle + \overline{r_2}\langle x,z\rangle$ □ $\underbrace{\overline{\langle r_1 y + r_2 z, x\rangle}}_{i)} = \underbrace{\overline{r_1\langle y,x\rangle + r_2\langle z,x\rangle}}_{1.d} = \overline{r_1\langle y,x\rangle} + \ldots = \overline{r_1}\cdot\overline{\langle y,x\rangle} + \ldots$ conj 性质

4. matrix norm 不仅需要满足所述3 property, 应更有 $\|AB\| \leq \|A\|\cdot\|B\|$ eg. Frobenius $\|A\|_F = \sqrt{\sum_i \sum_j A_{ij}^2}$
希望 connect matrix norm & vector norm, ∴ 研究 induced norm (与定理1 up. norm equiv) 证明 $\|I\|=1$
令 $\|\ \|_{(n)}$ $\|\ \|_{(m)}$ vec norm on $\mathbb{R}^n$ $\mathbb{R}^m$ matrix norm $\|\cdot\|$ induced by $\|\cdot\|_{(n)}$ $\|\cdot\|_{(m)}$ iff $\forall A\in\mathbb{R}^{m\times n}$, $x\in\mathbb{R}^n$ $\|Ax\|_{(m)} \leq \|A\|\cdot\|x\|_{(n)}$
eg. $\|A\| = \max_{\|x\|_{(n)}=1} \|Ax\|_{(m)}$ 为 induced norm (见定理)
NOTE 由定理 i) (cont + VS compact) $\forall A$ $\exists \|x_0\| = 1$ st. $\|Ax_0\| = \|A\|$ (即 $\|A\|$ attainable)

5. spectral radius $\rho(A) = \max\{|\lambda(A)|\}$ note $\|A\| \geq \|Ax\| = |\lambda|\cdot\|x\| = \rho(A)$, induce norm 下界 radius, 故有时用 $\rho(A)$ approx $\|A\|_2$
令 λ-x 对应 $|\lambda(A)|_{max}$, norm x st. $\|x\|=1$

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证 1，real symmetric A ∀eigval ∈ℝ

□ 求 $\underset{\text{complex inner prod}}{\lambda(\bar{x}^Tx)} = \bar{x}^T(\lambda x) = (\bar{x}^TA)x \overset{A\ \text{symmetric}}{=} (A\bar{x})^Tx \overset{\wedge}{=} (\bar{\lambda}\bar{x})^Tx = \bar{\lambda}(\bar{x}^Tx)$

$\therefore (\lambda-\bar{\lambda})(\underset{\langle x,x\rangle_{\mathbb{C}}\geq0,\ \text{若}=0\text{则}x=0,\ \text{但}x\ \text{evec}}{\bar{x}^Tx}) = 0$ $\therefore \lambda=\bar{\lambda}$

1.（Spectral Thm for symmetric A）

i）∀eigval $\lambda\in\mathbb{R}$ 且对应 eigvec real（证 1） 总有n个eigvec

ii）distinct eval 的 evec orthogonal（与P_{11}）

iii）∃ diag $D\in\mathbb{R}^{n\times n}$ & orthogonal $U\in\mathbb{R}^{n\times n}$ s.t. $A_{n\times n} = U_{n\times n}D_{n\times n}U^T_{n\times n}$，$D=\text{diag}(\lambda_{1\ldots n})$ U orthogonal

在 complex 上 Schur decomp → U unitary

称为 spectral decomposition □ 由 $\underset{22}{\text{四}}$.4 知 diagonalizable，因 $U=[v_1\ldots v_n]$，$U^TU=I$，$\therefore A=UDU^T=UDU^{-1}$

iv）$\{v_1\ldots v_n\}$ eigvec 为 orthogonal basis（即 $v_i^Tv_j\begin{cases}=0 & i\neq j\\ \neq0 & i=j\end{cases}$）

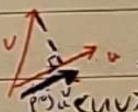
即使不一定有n distinct eigval □ U可逆，∴full-rank，∴L.Ind.$\{v_i\}$，可由 Gram Schmidt 生成 orthonormal

证 2 diagonalizable A 有 orthogonal evec ⇒ A symmetric

□ A 可对 $\overset{\boxed{25}.4}{A} = PD\underline{P^{-1}}$ $= P^T$，因 evec orthogonal $A^T = P^{-1T}D^TP^T = \underset{P^{-1}=P}{P^{T^{-1}}}DP^{-1} = A$

num stable 见 B6.3 例见 B7.例1

2.（Gram-Schmidt Process）



易知 $\text{proj}_u v$ 与u同方向 且 $v-\text{proj}_u v \perp u$

对 L.Ind $\{v_{1\ldots k}\}$ → orthonormal L.Ind $\{e_{1\ldots k}\}$. Def $\text{proj}_u(v) = \frac{\langle u,v\rangle}{\langle u,u\rangle}u$，$\text{proj}_0(v)\overset{\wedge}{=}0$

□ $u_1=v_1$ $e_1=\text{norm}(u_1)=\frac{u_1}{\|u_1\|}$

$u_2 = v_2 - \text{proj}_{u_1}(v_2)$ $e_2=\text{norm}(u_2)$

⋮

$u_k = v_k - \sum_{i=1}^{k-1}\text{proj}_{u_i}(v_k)$ $e_k=\text{norm}(u_k)$

Induction 证各自 orthogonal 即 each new $u_i \perp \forall u_j$ $j=1\ldots i-1$

eg. $u_1u_2 = v_1^Tv_2 - v_1^T\text{proj}$ $=0$

若 v_j 为 1st L.dep vector 则 $u_j=0$，break Alg

即 $u_k = v_k - \text{proj}_{\langle u_{1\ldots k-1}\rangle}(v_k)$ where proj 为 projection construct by P52.2 → $u_k = v_k - \sum\frac{\langle u_i,v_k\rangle}{\langle u_i,u_i\rangle}u_i = v_k - \sum(e_i^Tv_k)\cdot e_i$

证 3 A symmetric A^{-1} 也 symmetric □ $(A^{-1})^T = (A^T)^{-1} = A^{-1}$

2. 即使 $\begin{pmatrix}0&0\\0&0\end{pmatrix}$ symmetric 也有 spectral decomp 的 U（eg. natural basis I），因构建 by process on A

- 令 $\lambda_1\ v_1$ epair，find orthonormal $\{v_1\ u_{2\ldots n}\}$ basis（总可找到 L.Ind $u_{2\ldots n}$ wrt v_1，再 Gram Schmidt），令 $U_1=(v_1\ u_2\ldots u_n)$，note $U_1^TAU_1$ symmetric
- 1st col of $U_1^TAU_1$ 为 $(U_1^TA)v_1 = U_1^T\lambda_1v_1 = \lambda_1\begin{pmatrix}-v_1-\\-u_2-\\ \vdots\\-u_n-\end{pmatrix}\begin{pmatrix}|\\v_1\\|\end{pmatrix} = \begin{pmatrix}\lambda_1\\0\\ \vdots\\0\end{pmatrix}$ $\therefore U_1^TAU_1 = \begin{pmatrix}\lambda_1 & 0\\0 & B_{n-1,n-1}\end{pmatrix}$，因 B symmetric
- apply process on B .. until find orthonormal basis

证 4. $\{x_{1\ldots n}\}$ mutual orthogonal，$x_i\neq0$. 证 $\{x_{1\ldots n}\}$ L.Ind. □ 对 $\sum^n\alpha_ix_i=\vec{0} \xRightarrow{\text{乘}x_j\ j=1\ldots n} (\sum\alpha_ix_i)^Tx_j = \vec{0}^Tx_j$

$\alpha_jx_j^Tx_j = 0$

$\alpha_j=0$

证 5 对 symmetric A $\lambda_{\min}(A^{-1}) = \frac{1}{\lambda_{\max}(A)}$；$\lambda_{\max}(A^{-1}) = \frac{1}{\lambda_{\min}(A)}$

□ A eval ℝ 及 $\boxed{15}$.①

orthogonal & eigen4

Date. / [illegible] P55

T_0 $x\perp y$ 当 $\|x+y\|^2=\|x\|^2+\|y\|^2$ 展开 $\|x+y\|^2$

1、 x, y orthogonal iff $\langle x,y\rangle=0$ ‖ A orthogonal matrix iff $A^T=A^{-1}$ (即 $A^TA=I$)

Generalization of rotation P21 & reflection & permutation P54 ‖ 证 col of A 各自 orthogonal 且 norm 为 1 (orthonormal)

□ $\forall i,j$ $(A^TA)_{ij}=I_{ij}=0=\sum_k A_{ki}A_{kj}$ col dot product

$i=j$ 时 P_{ij} - =1 故行 row 也 orthonormal

(即 orthog basis if normalize each elem)

A 有 $v_i=|\lambda_i|$ [illegible] (全推 1)

2、1(a) 对 symmetric A, 1) 其 eigenval ∈ $\mathbb{R}$ 且 对 eigvec $v,\hat{v}$ associate with eval $\lambda\neq\hat{\lambda}$, $v,\hat{v}$ orthogonal

□ $\hat{v}^T(Av)=\lambda\hat{v}^Tv$, 又 $=(A^T\hat{v})^Tv=\hat{\lambda}\hat{v}^Tv$ ∴ 只当 $\hat{v}^Tv=0$ 时 $\lambda\hat{v}^Tv\equiv\hat{\lambda}\hat{v}^Tv$

1b) $A=\sum_{i=1}^{n}\lambda_iu_iu_i^T$, 当 $\{(\lambda_i,v_i)\}_{i=1}^n$ 为 eigen pairs, v_i normalized (1c)

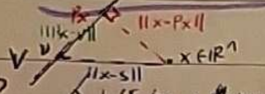
□ $AI=A=A\sum v_iv_i^T$, 考虑 $Av_i=\lambda_iv_i$, 同乘 v_i^T 并 sum over i, $\sum_i Av_iv_i^T=\sum_i\lambda_iv_iv_i^T=AI=A$

1c) $I=\sum^n v_iv_i^T$ $\{v_i\}^n$ eigen vec

□ $\{v_i\}^n$ orthogonal basis, ∴ 对 $\forall x=\sum_k^n\alpha_kv_k$, 考虑 $\boxed{\sum_i^n v_iv_i^T}x=\sum_i^n\sum_k^n v_iv_i^T\alpha_kv_k=\sum_k\alpha_kv_k=x$

$=\delta_{ik}$ Kronecker delta, 因 orthogonal 基

因 x 任意 . □ 左为 I



易证为 subspace, 即使 V 非 subspace

3、 令 VS 的 orthogonal complement $V^\perp=\{x\mid v^Tx=0,\forall v\in V\}$ (有 $V\oplus V^\perp=\mathbb{R}^n$, 当 V subspace)

易证 linear $T: VS\ \mathbb{R}^n\to V$

对 transform T orthogonal projector onto V, iff $\forall x\in\mathbb{R}^n$, $T(x)\in V$ 且 $x-T(x)\in V^\perp$

Th P orthogonal projector onto subspace $V=R(P)$ $\Longleftrightarrow$ $P^2=P=P^T$ (symmetric)

def, $x-Px\in R(P)^\perp$ $N(P^T)$

□ $\Rightarrow$ $R(I-P)\subseteq R(P)^\perp$, ∴ $\forall x\in V$ 有 $P^T(I-P)x=\vec{0}$ ∴ $P^T(I-P)=0$ zero matrix ∴ $P^T=P^TP$

∴ $P=P^{T^T}=(P^TP)^T=P^TP=P^T$ $\Leftarrow$ 考虑 $\forall x,y\in V$, $(Py)^T(I-P)x=y^TP(I-P)x=0$

∴ $(I-P)x\in R(P)^\perp$

2a) $R(A)^\perp=N(A^T)$ 且 $N(A)^\perp=R(A^T)$

□ 1° $\subseteq$ 设 $x\in R(A)^\perp$, $\forall v\in R(A)$, 则 $\exists y\in V$ $v=Ay$, 由 $(Ay)^Tx\overset{恒}{=}0$, $y^TA^Tx=0$, 即 $A^Tx=0$ $x\in N(A^T)$

$\supseteq$ 设 $x\in N(A^T)$ $\forall v\in R(A)$ ~~——~~, 考虑 $v^Tx=y^TA^Tx=0$

$L_A:\mathbb{R}^n\to\mathbb{R}^m$

$\mathbb{R}^n=N(A)\oplus R(A^T)$

$\mathbb{R}^m=N(A^T)\oplus R(A)$

2° apply ⊥ on both side 且 由 2c) 即 $N(A)$ vec ⊥ A^T col span vector

2b) $V^{\perp\perp}=V$ $\subseteq$ 令 $\{a_1,\dots a_n\}$ V, $\{b_1,\dots b_n\}$ $V^\perp$ basis, by def $V=R(A=[a_1\cdots a_n])$, $V^\perp=R(B=[b_1\cdots b_n])$ ∴ WTS $R(B)\subseteq R(A)$

设 $x\in N(A^T)$ $R(A)^\perp=V^\perp$ ∴ $\forall v\in V$ 有 $v^Tx=0$, 因 $Bv\in V$, $(Bv)^Tx=0=v^T(B^Tx)$, 因 v arbitrary, $B^Tx=0$, $x\in N(B^T)$ ∴ 由 四.3

$\supseteq$ 易证 by def

2c) $V\subseteq W\Rightarrow W^\perp\subseteq V^\perp$ □ $\forall w'\in W^\perp$ ∴ $\forall w\in W$ 有 $w^Tw'=0$ 而 $\forall v\in V$ 也 $\in W$

2d) P 可 orthogonal decomposition VS. $\forall x\in VS$, $\underset{\in V}{Px}+(I-P)x\in V^\perp$ 由 def

unique

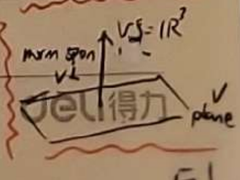
2e) P orthogonal on V, 则 $Px=x,\forall x\in V$ 且 $R(P)=V$ 且 $N(P)=V^\perp$

□ 1° x orthogonal decomp 为 $Px+(I-P)x$, 但 $x=x+0$ 也为 因 $x\in V$, ∴ $Px=x$ (orthogonal decomp unique)

2° by def $R(P)\subseteq V$ 但 $Px=x$, ∴ $V\subseteq R(P)$

3° 同取 ⊥ $N(P)^\perp=V^{\perp\perp}=V=R(P^T)=R(P)$

2f) $\forall x\in VS$, $\|Px\|\le\|x\|$ □ $\|x\|^2\overset{2d)}{=}\|Px+(I-P)x\|^2\overset{T_0}{=}\|Px\|+\dots\le\|Px\|$



orthocall 2.

Date.　　/

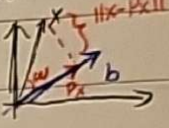
1. matrix A 為 projection matrix iff $A^2=A$ (idempotent)
即 $P^T=P=P^2$, 見L之3
若 $\langle Px,y\rangle=\langle x,Py\rangle$ 則為 orthogonal projection, 若不 orthogonal 則 oblique

將 $\begin{smallmatrix}x\\y\\z\end{smallmatrix}$ 投 $\begin{smallmatrix}x\\y\\0\end{smallmatrix}$ 垂直於 xy-plane 即為 $P=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$ 有 $P^2\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x\\y\\0\end{pmatrix}=P\begin{pmatrix}x\\y\\z\end{pmatrix}$

Pi) projection A eval = 0/1 因 $A^2v=\lambda^2v = Av=\lambda v$, ∴ $(\lambda^2-\lambda)\underline{v}=0$ (非0vec)

ii) proj A,B 乘不一定 proj 因 $\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1/2&1/2\\1/2&1/2\end{pmatrix}=\begin{pmatrix}1/2&1/2\\0&0\end{pmatrix}$
除非若 $AB=BA$　$(AB)^2=ABAB=AABB=A^2B^2=AB$

2. construct projection on 1° 1-D line span$\{\vec{b}\}$ of vector $\vec{x}$



- project $\vec{x}$ on line, ∴ $Px=\lambda\vec{b}$
- projected Px 與 x closest, 即 $\min\|x-Px\|$
只當垂直, ∴ $\langle Px-x,b\rangle=0=\langle\lambda b-x,b\rangle \Longleftrightarrow \lambda=\frac{b^Tx}{\|b\|^2}$
- ∴ projected $Px=\frac{b^Tx}{\|b\|^2}b$　Note 由此可知 $\|Px\|=\|\lambda b\|=|\lambda|\cdot\|b\|=\frac{|b^Tx|}{\|b\|^2}\|b\| = |\cos\omega|\|x\|\cdot\|b\|\cdot\frac{\|b\|}{\|b\|^2}$
$= |\cos\omega|\cdot\|x\|$

且 $= b\lambda=\frac{bb^T}{\|b\|^2}x$ ∴ $P=\frac{bb^T}{\|b\|^2}$　Pi): Px 為 P evec, eval = 1

2° Project on $V\subseteq\mathbb{R}^n$, $\dim V=m\geq1$ 有 basis $\{\vec{b}_{1..m}\}$

- project x on V, $Px=\sum^{m}\lambda_i\vec{b}_i=[\vec{b}_1\ldots\vec{b}_m]\begin{bmatrix}\lambda_1\\\vdots\\\lambda_m\end{bmatrix}=B\lambda$　m,n　n,1
- 垂直/orthogonal to all V's basis $\{b_{1..m}\}$, ∴ $\langle Px-x,b_1\rangle=0=b_1^T(x-B\lambda)$　$\Leftrightarrow\begin{pmatrix}-b_1-\\\vdots\\-b_m-\end{pmatrix}(x-B\lambda)=0$
$\vdots$
$\langle Px-x,b_m\rangle=0=b_m^T(x-B\lambda)$　$\Leftrightarrow B^T(x-B\lambda)=0$
$\Leftrightarrow B^TB\lambda=B^Tx$

∴ $\lambda=(B^TB)^{-1}B^Tx \overset{\text{pseudo-inverse}}{=} B^+x$

- ∴ projected $Px=B\lambda$, $P=B(B^TB)^{-1}B^T$ (當 m=1, B^TB scalar, reduce 為 1°)

例1. 1° 對 line $\left\langle\begin{pmatrix}1\\2\\2\end{pmatrix}\right\rangle$, projection $P=\frac{1}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}(1\,2\,2)=\frac{1}{9}\begin{pmatrix}1&2&2\\2&4&4\\2&4&4\end{pmatrix}$, note 對任意 x 如 $\begin{pmatrix}1\\1\\1\end{pmatrix}$ $P(x)=\frac{1}{9}\begin{pmatrix}5\\10\\10\end{pmatrix}$
且 $P^2(x)=P(x)$

2° 對 plane $U=\left\langle\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}0\\1\\2\end{pmatrix}\right\rangle$, $P=B(B^TB)^{-1}B^T=\begin{pmatrix}1&0\\1&1\\1&2\end{pmatrix}\ldots=\frac{1}{6}\begin{pmatrix}5&2&-1\\2&2&2\\-1&2&5\end{pmatrix}$, $P(x)=B\lambda=B\underset{\sim}{\begin{pmatrix}5\\-3\end{pmatrix}}$　$B^TB\lambda=B^Tx$

應用 least square 即 find $Ax=b$ 無 sol 時的近似 sol: find $\vec{x}^*\in Col(A)$ 使與 b closest
(orthogonal projection of b onto R(A)

quadratic form & norm 2 → P54.

Date. /

$\sum_{i,j} Q_{ij} x_i x_j$ 最高2次 ∴ quadratic

1. quadratic form is $f: \mathbb{R}^n \to \mathbb{R}$, $x \to x^T Q x$, $Q \in \mathbb{R}^{n\times n}$

NOTE Q 应 symmetric, 若非, 可用 $Q \leftarrow \frac{1}{2}(Q+Q^T)$ 代替, Q_{new} symmetric

对 quadratic form x^TQx $\begin{cases} \text{positive definite} & >0 \text{ for } x\neq 0 \\ \text{positive semidefinite} & \geq 0 \quad \forall x \\ \text{negative definite} & <0 \quad \forall x\neq 0 \\ \text{negative semidefinite} & \leq 0 \quad \forall x \end{cases}$

2. (Sylvester Criterion) x^TQx pos definite iff all leading ~~principal~~ minor of $Q>0$

det of Q successively rm last row & col

P54.3

证 1° 记 Δ_i ith leading prin. minor 设 $x = \sum^n x_i \vec{e}_i = \sum^n \tilde{x}_i \vec{v}_i$ ∴ $x = [v_1 \cdots v_n]\tilde{x}$, ∴ $x^TQx \overset{P54}{=}$

$\Delta_0 = 1$ natural basis of $\mathbb{R}^n$

$\tilde{x}^T V^T Q V \tilde{x} = \tilde{x}^T \bar{Q} \tilde{x}$ when $\bar{Q} = V^TQV$, note $\bar{q}_{ij} = \langle v_i, Qv_j\rangle$ 希望 $\bar{q}_{ij} = 0, i\neq j$

考虑 build basis v_i as $\left| \begin{array}{l} v_1 = \alpha_{11}e_1 \\ v_2 = \alpha_{21}e_1+\alpha_{22}e_2 \\ \vdots \\ v_n = \alpha_{n1}e_1+\dots+\alpha_{nn}e_n \end{array}\right.$

希望 pick $\{\alpha_{ij}\}$ st. 对 v_i 有以下: $\forall 1\leq j\leq i-1$, $\langle v_i, Qe_j\rangle = 0$ ($\Longrightarrow \langle v_i, Qv_j\rangle = 0$), 但 $\langle e_i, Qv_i\rangle = 1$

∴ sys on $\begin{cases} \alpha_{i1}q_{11}+\alpha_{i2}q_{12}+\dots+\alpha_{ii}q_{1i} = 0 \\ \alpha_{i1}q_{i-1,1}+\alpha_{i2}q_{i-1,2}+\dots+\alpha_{ii}q_{i-1,i} = 0 \\ \alpha_{i1}q_{i1}+\alpha_{i2}q_{i2}+\dots+\alpha_{ii}q_{ii} = 1 \end{cases}$

$\Rightarrow \begin{bmatrix} q_{11} & \cdots & q_{1i} \\ & \vdots & \\ q_{i1} & \cdots & q_{ii} \end{bmatrix}\begin{bmatrix} \alpha_{i1} \\ \vdots \\ \alpha_{ii} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$ 由 Cramer, $\alpha_{ii} = \frac{1}{\Delta_i}\det\begin{pmatrix} q_{11} & \cdots & q_{1,i-1} & 0 \\ & \vdots & & \\ q_{i1} & \cdots & q_{i,i-1} & 1 \end{pmatrix} \Downarrow \frac{\Delta_{i-1}}{\Delta_i}$

此时 $\bar{Q} = \begin{bmatrix} \alpha_{11} & & 0 \\ 0 & \ddots & \alpha_{nn} \end{bmatrix} = \begin{bmatrix} 1/\Delta_1 & & \\ & \ddots & \Delta_{n-1}/\Delta_n \end{bmatrix}$ ∴ 在 $\{v_i\}$ basis 化 quadratic x^TQx 为 $\frac{1}{\Delta_1}\tilde{x}_1^2 + \dots + \frac{\Delta_{n-1}}{\Delta_n}\tilde{x}_n^2$

2° $\Delta_i > 0 \Longleftrightarrow$ p.d. $\Longrightarrow$ sum square, >0 $\Longleftarrow$ 若 $\exists k$ st. $\Delta_k = 0 = \det Q_k = \det\begin{pmatrix} q_{11} & \cdots & q_{1k} \\ & & \vdots \\ & & q_{kk} \end{pmatrix}$

∴ $N(Q_k) \neq \{0\}$, 有 $u\neq 0$ st. $Q_k u = 0$. ∴ 对 $x = \begin{bmatrix} u \\ 0 \end{bmatrix}$ $x^TQx = u^TQ_ku = 0$, 但因 $x\neq 0$, pd 应 >0, ∴ 矛盾 ∴ $\Delta_i > 0$

Note 对 psd, psd ⟺ all principal minor ≥ 0, eg. $\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix}$ $\Delta_1 = 2$, $\Delta_2 = 0$, $\Delta_3 = 0$ 但 eval $0, -1.4, 1.4$ indefinite (Optim. P6 P_{20})

3. Rayleigh quotient $R_A(x) = \frac{x^TAx}{x^Tx}$, A symmetric

note i) $R_A(x) = R_A(\alpha x)$, $\alpha\neq 0$, $\vec{x}\neq\vec{0}$
ii) 当 x 为 A eigvec with eval λ, $R_A(x) = \lambda$

Thm 1 $\|x\|_2 = 1$ 有 $\lambda_{min} \leq x^TAx \leq \lambda_{max}$ λ wrt A

证 证 $\leq\lambda_{max}$ $A \overset{P50\ 1.}{=} UDU^T$, 令 $y = U^Tx$, $\max_{\|x\|=1} x^TAx \overset{y\leftrightarrow x\ 1\text{-}1,\ 因U orthogonal}{=} \boxed{\max_{\|y\|=1}} y^TDy = \max_{y_1^2+\dots+y_n^2=1} \sum^n \lambda_i y_i^2$

max 当 $i\in I = \{\underset{可能多个}{\text{indice}} \text{ of } \lambda_{max}\}$ $\sum_{i\in I} y_i^2 = 1$, $i\notin I$ $y_i = 0$, 此时 val $= \sum_{i\in I}\lambda_i y_i^2 = \lambda_{max}\sum_{i\in I} y_i^2 = \lambda_{max}$

由 i) 易得 (min-max) $\forall x\neq 0$ $\lambda_{min} \leq R_A(x) \leq \lambda_{max}$

4. 对 matrix induced norm, 可逆 A 有 $\|A^{-1}\| \geq (\|A\|)^{-1}$ 证 $\underbrace{\|AA^{-1}\|}_{I,\ P49.4性I} \overset{P49.4\ P(i)}{\leq} \|A\|\cdot\|A^{-1}\|$

in general $\|A^{-1}\| \gg$ RHS

Perturbutation Matrix

(col representation)

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Date.　　/

1. 对 permutation $\pi:\{1\cdots m\}\to\{1\cdots m\}$: $\begin{pmatrix}1 & 2 & \cdots & m\\ \pi(1) & \pi(2) & \cdots & \pi(m)\end{pmatrix}$ 有 permutation $P_{m\times m}$, $P_{ij}=\begin{cases}1 & \pi(i)=j\\ 0 & \neq\end{cases}$

$$P=\begin{bmatrix}-e_{\pi(1)}-\\ \vdots\\ -e_{\pi(m)}-\end{bmatrix}$$

e.g. 对 $\pi=\begin{pmatrix}1&2&3&4&5\\1&4&2&5&3\end{pmatrix}$ $P=\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&1&0&0\end{pmatrix}$

P i) Pv permutation 各 v_i, $Pv=\begin{bmatrix}v_{\pi(1)}\\ \vdots\\ v_{\pi(n)}\end{bmatrix}$ 　因 $v_i=\sum_k^n p_{ik}v_k=v_{\pi(i)}$

但 orthogonal 非 perm
因 orth 可[illegible] 于 SO, 1]

ii) P orthogonal, $\therefore P_\pi^{-1}=P_\pi^T=P_{\pi^{-1}}$ 　因 $P_\pi P_\pi^T$ 的 $ij=\sum_k P_{ik}P^T_{kj}=\sum_k P_{ik}P_{jk}=\delta_{ij}$ $\therefore P_\pi P_\pi^T=I_n$

iii) $P_\sigma P_\pi v=P_{\pi\circ\sigma}v$. $(\pi\circ\sigma)(i)=\pi(\sigma(i))$; $v^T P_\sigma P_\pi=v^T P_{\pi\circ\sigma}$

iv) v^TP permutate 各 v_i, $[v_1\cdots v_n]P=[v_{\pi^{-1}(1)}\ \cdots\ v_{\pi^{-1}(n)}]$

v) $\exists k$ s.t. $P_\pi^k=I$ 　因 每 $\pi\circ\pi\circ\cdots$ 总会走完 $\forall$ perm, k 约为 perm 的 order

vi) eval 为 unity $z^n=1$ 的 $n\in s$ (in unit complex circle) 　因 P orthogonal, $\therefore$ preserve norm 　由 P ii)

$\|Pv\|=\|v\|=\|\lambda v\|\Rightarrow|\lambda|=1$, 又有 $P^k=1$ $\therefore\lambda^k=1$ 　$\therefore\lambda=e^{i2\pi m/k}$, m=1 2 3.. Algebra P1

Unitary / Orthogonal 3 P51.1

Date. /

1. 对 matrix U, 称 Unitary iff $\underline{U}^H U = I_n = UU^H$ (若 real 称为 orthogonal; Hermitian / conjugate transpose)

P i) $|\det U| = 1$ 由 $T_1 . T_2$ ii) $\langle Ux, Uy\rangle = \langle x\ y\rangle$ (preserve complex inner product) $(Ux)^H Uy = x^H U^H U y$

iii) preserve L_2 vector norm, $\forall x, \|Ux\|_2 = \|x\|_2$ $x^T U^T U x = x^T x$ (x 可为 complex ∴ conjugate) 保持 angle 不变 $\cos\theta = \frac{x^T y}{\|x\|\|y\|}$ 在 $\begin{matrix} x \leftarrow Ux \\ y \leftarrow Uy \end{matrix}$ 下不变

iv) (Takagi Factorization) 对 symmetric A, $\exists$ unitary U, real ≥ 0 diagonal Σ st. $A = U\Sigma U^T$

U cols 为 orthonormal eigvec for $A\bar{A}$, Σ diagonal entry 为 square root of $A\bar{A}$ eigval

v) U unitary $\Leftrightarrow$ col orthonormal set □ $U^H U = [U^H u_1 \dots U^H u_n] = I_n$ 即 $(U^H u_j)_i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ $u_i^T u_j$

NOTE 由 vi) U unitary $\Leftrightarrow$ row orthonormal

vi) AB unitary 则 A^T, AB, A^{-1} unitary □ $(A^T)^T A^T = AA^T = 1$; $B^T A^T A B = 1$

T. U_{nn} unitary, 若 U triangular 且 diag entry > 0, $U = I$ (incidate)

假定上三 (由 P v))
□ u_1 为 norm 1 又 u_1 为 $\begin{pmatrix} \square \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ∴ $u_1 = e_1$; $u_2^T u_1 = 0$ 且 $\|u_2\| = 1$ 只有 $u_2 = e_2$ …

2. (non-square orthonormal set) 若 A_{mn} has col orthonormal, 则 $A^H A = I_n$

□ $(A^H A)_{ij} = \sum_k A^H_{ik} A_{kj} = \sum A_{ki} A_{kj} = \langle a_i, a_j\rangle = \delta_{ij}$

3. eval of unitary U in complex unit circle. □ P iii) $\|U\cancel{x}v\| = |\lambda| \|v\|$, $= \|v\|$ thus $|\lambda| = 1$

4. Q_{mn} col orthonormal note 只有 $m \geq n$, 此时 SVD $V = I_{nn}$, $\Sigma = I_{mn}$, $\cancel{U} = (Q | \dots)$

□ $m < n$ 时不可能有此 Q 因 col of Q in $\mathbb{R}^m$ 上不会有比 m 多的 orthonormal

∴ 由 $Q^T Q = I_n$ (T_2) $\sigma_i = 1$, ∴ $\Sigma = I_{mn}$, $V = I_n$ 则 $Q_{mn} = U_{mm} I_{mn} I_n = (Q_{mn} | u_{n+1} \dots u_m)$

where u_i orthonormal. note 此 u_i 必可 因可找到 $\{u_{n+1 \dots m}\}$ L.ind 再 Gram Schmidt

5. $\|U\|_2 = 1$ □ $\|U\|_2 = \sup_{x \neq 0} \frac{\|Ux\|_2}{\|x\|_2} = 1$ (P iii))

6. (preserve matrix L_2 norm) $\|AU\|_2 = \sup_{\|x\|=1} \|AUx\| \overset{P iii)}{=} \sup_{\|Ux\|=1} \|AUx\| \overset{y \leftarrow Ux}{=} \sup_{\|y\|=1} \|Ay\| = \|A\|_2$

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Date. /

例1 Rank分解 $A=\begin{pmatrix}1&3&1&4\\2&7&3&9\\1&5&3&1\\1&2&0&8\end{pmatrix} \sim \overset{\text{RREF}}{\begin{pmatrix}1&0&-2&0\\0&1&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}}$ 口 $C=\begin{pmatrix}1&3&4\\2&7&9\\1&5&1\\1&2&8\end{pmatrix}$，$F=\begin{pmatrix}1&0&\overset{a_3=-2a_1+a_2}{-2}&0\\0&1&1&0\\0&0&0&1\end{pmatrix}$

例2 QR分解 $A=\begin{pmatrix}12&-51&4\\6&167&-68\\-4&24&-41\end{pmatrix}$ 口 col l.ind. 用陪0.2 Gram Process $\rightarrow \begin{bmatrix}12&-69&-58/5\\6&158&6/5\\-4&30&-33\end{bmatrix} \rightarrow \begin{bmatrix}-6/7&-69/175&-58/175\\3/7&158/175&6/175\\-2/7&6/35&-33/35\end{bmatrix}$

$v_1 \quad v_2 \quad v_3$ $\qquad Q = q_1 \quad q_2 \quad q_3$

$Q^TA=Q^TQR=R=\begin{pmatrix}14&21&-14\\0&175&-70\\0&0&35\end{pmatrix}$

例3 用QR解 sys eq. $A\vec{x}=\vec{b}$. $\quad QR\vec{x}=\vec{b} \overset{\text{同左乘}Q^T}{\iff} R\vec{x}=Q^T\vec{b}$，因 R upper △，用 backward sub

解 $\begin{pmatrix}r_{11}x_1+\dots+r_{1n}x_n\\r_{22}x_2+\dots+r_{2n}x_n\\\vdots\\r_{nn}x_n\end{pmatrix}=Q^T\vec{b}$

Decomposition

57

Date. /

1 对 A_{mn} rk=r | rank decomposition | 为 $A_{mn} = \overset{\text{full-col-rank}}{C_{mr}} F_{rn}$ full-row-rank

rk=r, R(A)=span{$a_{1..n}$} →

令 A 中 c.Ind 的 r 个 colvector $\vec{c}_{1..r}$, $C=[\vec{c}_1 .. \vec{c}_r]$, 而 $\forall j = r+1..n$ 有 $\vec{a}_j$ 被表为 r 个 L.comb, $F = \begin{pmatrix} 1 & & c_{1,r+1} \cdots c_{1n} \\ & \ddots & \vdots \\ & 1 & c_{r,r+1} \cdots c_{rn} \end{pmatrix}$

$(\vec{a}_j = \sum_{i=1}^{r} f_{ij} \vec{c}_i)$

$P_{1a)}$ not unique. 对 ∀ 可逆 R, $C \leftarrow CR$, $F \leftarrow R^{-1}F$ 也为 rk decomposition [22]. P_{1a}

b) $\begin{cases} m<n & \text{rk}A=m : A = I_m A \\ > & n : A = A I_n \end{cases}$

2. $m \geq n$ A_{mn} $\text{rk} = n$ (col L.Ind, col full-rank) | QR decomposition | 为 $A = \overset{\text{unitary}}{Q_{mm}} R_{mn} \xrightarrow{\text{upper triangular}} \begin{bmatrix} \bar{R}_{nn} \\ 0 \end{bmatrix}$ 且 $= \bar{Q}\bar{R}$

Q 为 A 的 L.Ind col {a_i} Gram-Schmidt Process 生成 的 orthonormal $[q_1 .. q_n]$, 因 $\forall k=1..n$ 有 $a_k = r_{k1} q_1 + \dots + r_{k,k-1} q_{k-1} + r_{kk} q_k$ P50 笔

$q_i^T v_k = \vec{q}_i^T a_k$ $\|v_k\| = \|a_k\|$

∴ $\bar{Q}_{mn} = (q_1 .. q_n)$ $\bar{R} = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ 0 & \ddots & r_{nn} \end{pmatrix}$ 易知 $Q^T Q = I_n$, $\bar{R}$ entry ≥ 0

$Q = (\bar{Q} \mid Q')$ pick st Q orthogonal

$P_{2a)}$ 若 A square (m=n ⇒ invertible), R 也可逆. ∵ $\det A = \det Q \det R$ ($P_1 T_2$ ±1 P35 P_1) ∴ $|\det A| = |\det R| \neq 0$

2b) QR 不 unique ???

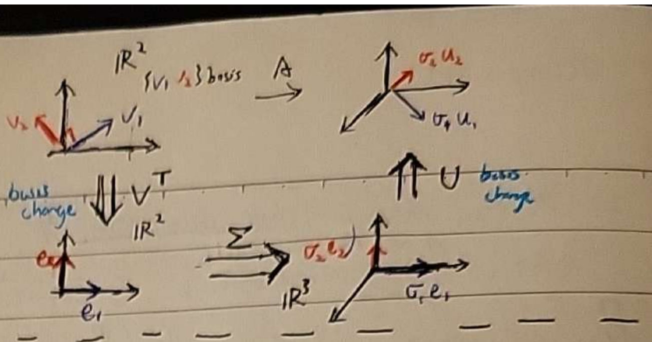
3. LU decomposition ***5.应用

4. UV decomposition ***5.*

5. 对 pd cholesky / LDL^T 分解. P_1

Date.　　/

例1. SVD 几何意义 ；对 $A \in \mathbb{R}^{3\times 2}$



对 linear $T: \mathbb{R}^n \to \mathbb{R}^m$

B C 为 $\mathbb{R}^n$ $\mathbb{R}^m$ standard basis

$\bar{B}$ $\bar{C}$ 〃〃 任意 basis

- V 为 $Q_{\bar{B}\to B}$ ∴ $V^T = V^{-1} = Q_{B\to\bar{B}}$，将 $[x]_B$ 转换为 $\bar{B}$ representation
- Σ scale each dim by σ_i 并 ~~AA~~/增 dim 成 $\mathbb{R}^m$（$\bar{C}$ 表示）
- U 为 $Q_{\bar{C}\to C}$，令 output of ~~T~~(x) 处于 $[T(x)]_C$

应用1 A $\overset{\text{user}}{\text{movie rating}}$ 　若 i) all user rate use linear

ii) rating 无 noise

iii) u_i typical movie

v_i typical viewer

例2. SVD $A = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 4 & & & 0 \\ & 2 & & 0 \\ & & \underset{\boxed{19}}{0} & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \\ 1/\sqrt{2} & 1/2 & 1/2 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$ 　找 basis for $N(A)$, $R(A)$

] 1° U orthogonal，∴可逆，∴bijection ∴ $Ux=0 \Leftrightarrow x=0$ ∴只用考虑 x s.t. $\Sigma V^T x = 0$ $\boxed{17}$.3.ii)

为简单，考虑 y s.t. $\Sigma y = 0$，y 应 $\begin{pmatrix} 0 \\ 0 \\ * \\ * \end{pmatrix}$ 即 应令 x 满足 $V^T x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ 或 $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\xRightarrow{V^T \text{ orthogonal}}$ $x = V\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = V$ 3rd col / 4rd col

2° V^T bijection ∴只用考虑 $U\Sigma$ 如何 map input $x\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ 因 $\Sigma x = \begin{pmatrix} 4a \\ 2b \\ 0 \end{pmatrix}$ ∴ $U\Sigma x$ 为 $\begin{pmatrix} 4a/\sqrt{2} - 2b/\sqrt{3} \\ 4a/\sqrt{2} + 2b/\sqrt{3} \\ 2b/\sqrt{3} \end{pmatrix}$

$= \begin{pmatrix} 4/\sqrt{2} \\ 4/\sqrt{2} \\ 0 \end{pmatrix} a + \begin{pmatrix} -2/\sqrt{3} \\ 2/\sqrt{3} \\ 2/\sqrt{3} \end{pmatrix} b$，$R = \left\langle \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \right\rangle$

Decomposition 2

$m\boxed{A}^{n} = \boxed{U}\boxed{\Sigma}\boxed{V^T}$ Σ为 $\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_n \\ 0 \cdots & 0 \\ 0 \cdots & 0\end{pmatrix}$ or $\begin{pmatrix}\sigma_1 & 0 & 0\cdots 0\\ 0 & \sigma_n & 0\cdots 0\end{pmatrix}$

Date.

1. SVD $\boxed{\text{any}}$ $A_{mn} = \overset{\text{orthogonal}}{U_{mm}\Sigma_{mn}V_{nn}^T}$ 令 rk A = r $\sigma_1 \geq \dots \geq \sigma_r > \sigma_{r+1} = \dots = \sigma_{\min(n,m)} = 0$

见.5.t1 diagonal with singular val σ_i, $\sigma_i^2(A) = \lambda_i(A^TA) \geq 0$ 因 A^TA psd optim. P6.

∴ $A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$ 即 Sum of rk-1 matrix 4.T3

2. 构建 SVD:

1° A^TA symmetric 且 psd (P6. P(1)), ∴ $\underbrace{\text{eigen diagolize}}$ (Psu.1) 为 $PDP^{-1} = P\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}P^T$, $\lambda_i \overset{\text{psd}}{\geq} 0$

但若 SVD ∃, $A^TA = V\Sigma^TU^TU\Sigma V^T = V\begin{bmatrix}\sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2\end{bmatrix}V^T \rightarrow \begin{cases} V^T = P^T \\ \sigma_i^2 = \lambda_i \end{cases}$

∴ A^TA 的 eigenvec (orthonormal by Spectral Thm) 为 V^T row / V col

即 $v_i \perp v_j$

2° $\{v_i\}$ orthonormal, 而 $\{Av_i\}_{i=1}^{r}$ 在 $m \geq r$ 下 basis 且 $(Av_i)^T(Av_j) = v_i^T(A^TA)v_j = v_i^T\lambda_j v_j = 0$

orthogonal basis

3° 因此用 $\{Av_i\}^r$ 构建 U, 为 orthonormal $u_i = \frac{Av_i}{\|Av_i\|} = \frac{Av_i}{\sqrt{v_i^TA^TAv_i}} = \frac{Av_i}{\sqrt{\lambda_i}\|v_i\|} = \frac{Av_i}{\sigma_i}$ $i = 1..r$

(singular val eq)

u_i 为 AA^T evec: $AA^Tu_i = \frac{A\cdot(A^TAv_i)}{\sigma_i}$

4° 对 remaining m−r, find L.I. vec wrt $u_{1..r}$ 并 Gram-Schmidt orthonormalize

3. P(1) orthogonal Q 有

Matrix calculus

matrix calculus, symbolic calculus

Date. $f: \mathbb{R} \to \mathbb{R}^n$, $\frac{\partial \vec{f}}{\partial x} = \begin{bmatrix} \partial f_1(x)/\partial x \\ \vdots \\ \partial f_n(x)/\partial x \end{bmatrix}$ 也称 numerator layout　denominator layout 为 transpose　记为 $\frac{\partial f}{\partial \vec{x}}$

1. 对 $f: \mathbb{R}^n \to \mathbb{R}$, $(x_1, x_2, \dots x_n) \to f(x_1 \dots x_n)$, 有 $\nabla f^T = Df = \left[\frac{\partial f}{\partial x_1} \dots \frac{\partial f}{\partial x_n}\right]$ $1 \times n$　grad　total differential

而对 $f: \mathbb{R}^n - \mathbb{R}^m$, $(x_1 \dots x_n) \to \begin{bmatrix} f_1(x_1 \dots x_n) \\ f_m(x_1 \dots x_n) \end{bmatrix}$ 有 Jacobian $J = \begin{bmatrix} Df_1 \\ Df_m \end{bmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ & \vdots & \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$ $m \times n$

称 J 为 $\frac{\partial f(x)}{\partial x}$ (vec by vec)

2. ✱ $\frac{\partial(\vec{x} + \vec{y})}{\partial x} = I$, $\frac{\partial(x+y)}{\partial y} = I$　$x_1 + y_1$　where $x\, y \in \mathbb{R}^n$

□ $\frac{\partial(x+y)}{\partial x} = \begin{pmatrix} \frac{\partial}{\partial x_1}(f_1(x)+g_1(y)) & \dots & \partial/\partial x_n \underbrace{(f_1(x)+g_1(y))} \\ & \vdots & \\ \partial/\partial x_1 (f_n(x)+g_n(y)) & \dots & \partial/\partial x_n (f_n(x)+g_n(y)) \end{pmatrix} = \begin{pmatrix} 1 & \ddots & 0 \\ & & 1 \end{pmatrix} = I$

x+y 视为 f(x)+g(y)
f, g identity 如右例 1

✱ $\frac{\partial(x-y)}{\partial x} = I$, $\frac{\partial(x-y)}{\partial y} = -I$

✱ $\boxed{f: \mathbb{R}^n \to \mathbb{R}^n,\ x \to f(x),\ \frac{\partial f(x)}{\partial x} = \mathrm{diag}(f'(x))}$ elementwise

□ $\frac{\partial f(x)_i}{\partial x_j} = \begin{cases} f'(x_i), & i=j \\ 0 \end{cases}$

hadamard

✱ $\frac{\partial(x \odot y)}{\partial x} = \mathrm{diag}(y)$, x, y, $\frac{\partial(x \circ y)}{\partial y} = \mathrm{diag}(x)$

□ $\frac{\partial(x \circ y)}{\partial x} = \begin{pmatrix} \partial/\partial x_1 \underbrace{(f_1(x) * g_1(y))} & \dots \\ \vdots & \\ \partial/\partial x_1 \underbrace{(f_n(x) \cdot g_n(y))}_{x_n y_n} & \dots \end{pmatrix} = \begin{pmatrix} y_1 & \ddots & 0 \\ 0 & & y_n \end{pmatrix} = \mathrm{diag}(y_1 \dots y_n)$　0

✱ $\frac{\partial(x/y)}{\partial x} = \mathrm{diag}(\dots \frac{1}{y_i} \dots)$　$\frac{\partial(x/y)}{\partial y} = \mathrm{diag}(\dots, -\frac{x_i}{y_i^2} \dots)$

✱ note 对 scalar k, $\frac{\partial(k+x)}{\partial x} = \frac{\partial(k\vec{1}+x)}{\partial x} = I$　vector of 1, in $\mathbb{R}^n$, 即 $(1, 1, \dots, 1)$

同理 $\frac{\partial(kx)}{\partial x} = kI$, $\frac{\partial(kx)}{\partial k} = \left(\frac{\partial}{\partial k} k x_1, \dots, \partial/\partial k\, k x_n\right) = x$　vec by scalar, size $\mathbb{R}^n$

$\frac{\partial(k+x)}{\partial k} = \vec{1}$

3. chain Rule.

✱ 对 scalar x, $g: \mathbb{R} \to \mathbb{R}^k$, $f: \mathbb{R}^k \to \mathbb{R}^m$ 考虑 $y = f(g(x))$, $\frac{\partial y}{\partial x}$

$$\begin{bmatrix} \frac{\partial f_1(g(x))}{\partial x} \\ \vdots \\ \frac{\partial f_m(g(x))}{\partial x} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{k} \frac{\partial f_1}{\partial g_i} \frac{\partial g_i}{\partial x} \\ \vdots \\ \sum^{k} \frac{\partial f_m}{\partial g_i} \cdot \frac{\partial g_i}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial g_1} & \ddots & \frac{\partial f_1}{\partial g_k} \\ \frac{\partial f_m}{\partial g_1} & \dots & \frac{\partial f_m}{\partial g_k} \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial x} \\ \vdots \\ \frac{\partial g_k}{\partial x} \end{bmatrix} = \underline{\frac{\partial f}{\partial g}} \times \frac{\partial g}{\partial x}$$

multivar chain　Jacobian of g, x　Jacobian of f, g

✱ 对 vector $x \in \mathbb{R}^n$, $g: \mathbb{R}^n \to \mathbb{R}^k$　$f: \mathbb{R}^k \to \mathbb{R}^m$ 考虑 $\frac{\partial f(g(\vec{x}))}{\partial x}$

□ $J = \begin{bmatrix} \nabla f_1 \\ \vdots \\ \nabla f_m \end{bmatrix} = \begin{pmatrix} \frac{\partial f_1(g(\vec{x}))}{\partial x_1} & \dots & \frac{\partial f_1(g(\vec{x}))}{\partial x_n} \\ & \vdots & \\ \frac{\partial f_m(g(\vec{x}))}{\partial x_1} & \dots & \frac{\partial f_m(g(\vec{x}))}{\partial x_n} \end{pmatrix}$, $(i, j) = \underbrace{\frac{\partial f_i(g(\vec{x}))}{\partial x_j}}_{\text{vec}(\mathbb{R}^m) \text{ by scalar}} = \sum_{x=1}^{k} \frac{\partial f_i}{\partial g_x} \frac{\partial g_x}{\partial x_j} = \sum_x \left(\frac{\partial f}{\partial g}\right)_{ix} \left(\frac{\partial g}{\partial x}\right)_{xj}$

$$\therefore \frac{\partial f(g(\vec{x}))}{\partial \vec{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial g_1} & \dots & \frac{\partial f_1}{\partial g_k} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial g_1} & & \frac{\partial f_m}{\partial g_k} \end{pmatrix} \times \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \ddots & \frac{\partial g_1}{\partial x_n} \\ & \vdots & \\ \frac{\partial g_k}{\partial x_1} & \dots & \frac{\partial g_k}{\partial x_n} \end{pmatrix} = \underset{m \times k}{\frac{\partial f}{\partial g}} \cdot \underset{k \times n}{\frac{\partial g}{\partial x}}$$

NOTE 当 f_i 只与 g_i 有关, $\frac{\partial f}{\partial g} = \mathrm{diag}\left(\frac{\partial f_i}{\partial g_i}\right)$
$g_i \cdots x_i$　$\frac{\partial g}{\partial x} = \mathrm{diag}\left(\frac{\partial g_i}{\partial x_i}\right)$

thus $\frac{\partial}{\partial x} f(g(x)) = \mathrm{diag}\left(\frac{\partial f_i}{\partial g_i}\right) \cdot \mathrm{diag}\left(\frac{\partial g_i}{\partial x_i}\right)$
$= \mathrm{diag}\left(\frac{\partial f_i}{\partial g_i} \cdot \frac{\partial g_i}{\partial x_i}\right)$

例1 $f: \mathbb{R}^n \to \mathbb{R}^n$, $\vec{x} \to \vec{x}$, 即 $f_i(\vec{x}) = x_i$, 求 J ①

口 $J = \begin{pmatrix} \partial f_1(\vec{x})/\partial x_1 & \cdots & \partial f_1(\vec{x})/\partial x_n \\ & \vdots & \\ \partial f_n(\vec{x})/\partial x_1 & \cdots & \partial f_n(\vec{x})/\partial x_n \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I$

例2 $y = \sum_{i=1}^{n} x_i$ (scalar), 证 $\frac{\partial y}{\partial x} = \vec{1}^T$

口 $\frac{\partial y}{\partial x} = \left[\frac{\partial}{\partial x_1} \sum f_i(\vec{x}), \ldots, \frac{\partial}{\partial x_n} \sum_i f_i(\vec{x})\right] = [1, \ldots, 1] = \vec{1}^T$ ⟵ $\sum_i \frac{\partial f_i(\vec{x})}{\partial x_n} = \sum_i \frac{\partial x_i}{\partial x_n}$

size 同 x, vec 令 $y = \sum_{i=1}^{n} f(\vec{x})$, $f_i(\vec{x}) = x_i$

若 $y = k \sum^{n} x_i$, k scalar, $\frac{\partial y}{\partial x} = [k, \ldots k]$, $\frac{\partial y}{\partial k} = \sum^{n} x_i$

口 $\frac{\partial y}{\partial x} = [\underbrace{\frac{\partial}{\partial x_1} \sum f_i(\vec{x}) \cdot k}_{\sum_i \frac{\partial x_i k}{\partial x_1} = k}, \ldots]$, $\frac{\partial y}{\partial k} = \frac{\partial}{\partial k} \sum x_i k = \sum x_i$

例3 $\varphi(\vec{x}) = \text{Relu}(w^T x + b)$ where $w, x \in \mathbb{R}^n$, 求 $\frac{\partial \varphi(\vec{x})}{\partial w}$, $\frac{\partial \varphi(\vec{x})}{\partial b}$

口 对 $w^T x = \overset{\text{scalar}}{\underline{y}}$, 法1 $\frac{\partial y}{\partial w} = \left(\frac{\partial y}{\partial w_1} \cdots \frac{\partial y}{\partial w_n}\right) = (x_1 \cdots x_n) = x^T$

$\frac{\partial}{\partial w_n} \sum w_i x_i = \sum \frac{\partial}{\partial w_n} w_i x_i = x_n$

法2 $u = w \odot x$
$y = \text{sum}(u)$ $\frac{\partial y}{\partial w} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial w} = \overset{见例2}{\underline{\vec{1}^T}} \cdot \text{diag}(x) = x^T$

∴ 对 $z = y + b$, $\frac{\partial z}{\partial w} = \frac{\partial y}{\partial w} + \frac{\partial b}{\partial w} = x^T + 0^T = x^T$; $\frac{\partial z}{\partial b} = 0 + 1 = \overset{\text{scalar}}{1}$

∴ $\frac{\partial \varphi(\vec{x})}{\partial z} = \frac{\partial \max(0, z)}{\partial z} = \begin{cases} 0 & z \leq 0 \\ \frac{dz}{dz} = 1 & z > 0 \end{cases}$

∴ $\frac{\partial \varphi(x)}{\partial w} = \frac{\partial \varphi}{\partial z} \frac{\partial z}{\partial w} = \begin{cases} \vec{0}^T & z \leq 0 \\ x^T & z > 0 \end{cases}$ $\frac{\partial \varphi(x)}{\partial b} = \begin{cases} 0 & z \leq 0 \\ 1 & z > 0 \end{cases}$

例4.1 $\frac{\partial A\vec{x}}{\partial \vec{x}} = A$, $\vec{x} \in \mathbb{R}^n$, $A: \mathbb{R}^{m \times n}$

$(Ax)_i$ 即 ith row

口 令 $f: \mathbb{R}^n \to \mathbb{R}^m$, $x \to Ax$, $\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1(Ax)}{\partial x_1}, & \cdots, & \frac{\partial f_1(Ax)}{\partial x_n} \\ & \vdots & \\ & & \frac{\partial f_m(Ax)}{\partial x_n} \end{pmatrix}$, $\frac{\partial \boxed{f_i(Ax)}}{\partial x_j} = \frac{\partial \sum^{n} A_{ik} x_k}{\partial x_j} = A_{ij}$

thus $\frac{\partial f}{\partial x} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{m1} & & \end{pmatrix} = A$

2. $\alpha = \underset{(1,m)}{\vec{y}^T} A \underset{(n,1)}{\vec{x}}$ $\frac{\partial \alpha}{\partial x} = y^T A$, $\frac{\partial \alpha}{\partial y} = x^T A^T$

口 令 $w^T = y^T A$, $\alpha = w^T x$, by $\frac{\partial Ax}{\partial x} = A$, $\frac{\partial \alpha}{\partial x} = w^T$; 因 α scalar, $\alpha = \alpha^T = x^T A^T y$. 同理有 $\frac{\partial \alpha}{\partial y} = x^T A^T$

3. quadratic form $\alpha = \vec{x}^T A \vec{x}$, $x \in \mathbb{R}^n$ $A \in \mathbb{R}^{n \times n}$ $\frac{\partial \alpha}{\partial x} = x^T (A + A^T)$ 见③例2

口 $\alpha = \sum_i^n \sum_k^n x_k A_{ik} x_i$, $\frac{\partial \alpha}{\partial x} = \left(\frac{\partial \alpha}{\partial x_1} \cdots \frac{\partial \alpha}{\partial x_n}\right) = (\ldots, \underbrace{\overset{(x^T A)_j}{\sum_{k=1}^{n} x_k A_{jk}} + \overset{(x^T A^T)_j}{\sum_{i=1}^{n} x_i A_{ij}}}_{j\text{th}}, \ldots) = x^T A + x^T A^T$

$= x^T (A + A^T)$ 当 A symmetric 有 $2x^T A$

当 $A = I$, $\frac{\partial x^T x}{\partial x} = 2x^T$

4. $\frac{\partial x^T A}{\partial x} = A^T$, $x \in \mathbb{R}^m$ $A \in \mathbb{R}^{m \times n}$

口 $\begin{pmatrix} \frac{\partial (x^T A)_1}{\partial x_1} & \cdots & \\ \frac{\partial (x^T A)_n}{\partial x_1} & \cdots & \frac{\partial (x^T A)_n}{\partial x_m} \end{pmatrix}$ $\frac{\partial (x^T A)_i}{\partial x_j} = \frac{\partial}{\partial x_j} \sum_{k=1}^{m} x_k A_{ki} = A_{ji}$ $\therefore = \begin{pmatrix} A_{11} & \cdots & A_{n1} \\ A_{1n} & \cdots & A_{nn} \end{pmatrix} = A^T$

5. $\vec{x}, \vec{y} \in \mathbb{R}^n$ $\frac{\partial \vec{y}^T \vec{x}}{\partial \vec{x}} = \vec{y}^T$, $\frac{\partial \vec{y}^T \vec{x}}{\partial \vec{y}} = \vec{x}^T$ 口 $\vec{y}^T \vec{x} = \sum^{n} x_i y_i$, thus 分别为 $(y_1 \cdots y_n)$ $(x_1 \cdots x_n)$

6. $A \in \mathbb{R}^{n \times n}$ $\frac{\partial A^{-1}}{\partial \alpha} = -A^{-1} \frac{\partial A}{\partial \alpha} A^{-1}$

口 $A^{-1} A = I$ $\xRightarrow[\text{对}\alpha\text{求导}]{}$ $A^{-1} \frac{\partial A}{\partial \alpha} + \frac{\partial A^{-1}}{\partial \alpha} A = 0$

7. $\frac{\partial \vec{x}^T \vec{x}}{\partial \vec{x}} = 2\vec{x}^T$ 记忆为 $\frac{\partial y^T x}{\partial z} = x^T \frac{\partial y}{\partial z} + y^T \frac{\partial x}{\partial z}$ 4号例

口 $\frac{\partial \vec{x}^T x}{\partial \vec{x}} + \frac{\partial x^T \vec{x}}{\partial \vec{x}} = \vec{x}^T + \vec{x}^T$

1. 对 $\frac{\partial L}{\partial \underline{W}_{m,n}}$，可记为 $\frac{\partial L}{\partial \underline{\vec{W}}_{\text{flatten, } m\times n}}$，即 $\left[\frac{\partial L}{\partial W_1} \cdots \frac{\partial L}{\partial W_{mn}}\right]$，可以方便记为 $\begin{bmatrix} \frac{\partial L}{\partial W_{11}} & \cdots & \\ \frac{\partial L}{\partial W_{m1}} & \cdots & \frac{\partial L}{\partial W_{mn}}\end{bmatrix}$，即 repack ele in matrix form

mixed layout，若用 nominator，应当为 n×m

例1　对 $\overset{h}{Z} = \overset{1,d}{X}\overset{d,h}{W}$，求 $\frac{\partial L}{\partial W} = X^T\frac{\partial L}{\partial z}$

证 令 $\frac{\partial L}{\partial z} = \delta$，考虑 $\frac{\partial z}{\partial W}$，若用1. 则为 h×d×h tensor；∴ 只考虑 $\frac{\partial z}{\partial W_{ij}} = \begin{bmatrix}\partial z_1/\partial W_{ij}\\ \vdots \\ \partial z_h/\partial W_{ij}\end{bmatrix}$，而 $\frac{\partial z_k}{\partial W_{ij}} = \begin{cases} x_i, & i=l, j=k \\ 0 \end{cases}$　$\sum_l^d x_l W_{lk}$

∴ $\frac{\partial z}{\partial W_{ij}} = \begin{pmatrix}0\\ \vdots\\ x_i\\ \vdots\\ 0\end{pmatrix} k=j$．则 $\frac{\partial L}{\partial W_{ij}} \overset{上述了}{=} \delta \frac{\partial z}{\partial W_{ij}} = \sum_{k=1}^{h} \delta_k \frac{\partial z_k}{\partial W_{ij}} = \delta_j x_i$，∴ $\frac{\partial L}{\partial W} = \begin{pmatrix} x_1\delta_1 & \cdots & x_1\delta_h \\ \vdots & & \vdots \\ x_d\delta_1 & \cdots & x_d\delta_h\end{pmatrix} = X^T\frac{\partial L}{\partial z}$

2. 对 scalar func. f，对 scalar x 有 $df = f'(x)dx$

T_1　对 matrix/vec $\vec{x}$，有 $df = tr(\frac{\partial f}{\partial x} dx)$，dx shape 同 x，如 x matrix，$dx = \begin{pmatrix} dx_{11} & \cdots & dx_{1n} \\ \vdots & & \vdots \\ dx_{m1} & \cdots & dx_{mn}\end{pmatrix}$

证 若为 matrix　LHS $= \sum_{i,j} \frac{\partial f}{\partial x_{ij}} dx_{ij}$，RHS: $tr(\frac{\partial f}{\partial x}^{T^T} dx) = \sum_{i,j} (\frac{\partial f}{\partial x}^T)_{ij} dx_{ij}$　$(\frac{\partial f}{\partial x})_{ji} = \frac{\partial f}{\partial x_{ij}}$

(n,m)，strict nominator layout 非 1. 的 mixed layout

P. i) $d\,tr A = tr(dA)$　证 $tr\,dA = \sum_i dA_{ii} = d(\sum A_{ii}) = d\,tr A$

ii) (linearity) $d(A+cB) = dA + c\,dB$　由 matrix add linearity

iii) $d(AB) = dA\cdot B + A\cdot dB$　证 $LHS_{ij} = d(\sum_k A_{ik}B_{kj}) = \sum_k (dA_{ik}B_{kj} + A_{ik}dB_{kj})$

例2　上述 T_1 提供计算 $\frac{\partial f}{\partial x}$ 的另一方法 (trace)，$f(x) = \vec{x}^T A\vec{x}$，求 $\frac{\partial f}{\partial x}$ (上与例4.3)

证 $df = d\,tr(\overset{\text{scalar, } tr(s)=s}{x^TAx}) \overset{P\,ii)}{=} tr(d(x^TAx)) = tr(dx^T\cdot Ax + x^Td(Ax)) = tr(dx^T\cdot Ax + x^T\cdot \overset{\text{constant, } \therefore =0}{dA}\cdot x + x^TA\cdot dx) \overset{P.1}{=} tr(dx^TAx) + tr(x^TA\cdot dx)$

cyclic & combine $= tr(x^TA^Tdx + x^TAdx) = tr[(x^T\cdot(A^T+A))\cdot dx]$，∴ 由 T_1 $\frac{\partial f}{\partial x} = x^T(A+A^T)$

例3　证 $\frac{\partial \det A}{\partial A} = C = \text{adj}A^T$　证 $\frac{\partial \sum_j A_{ij}C_{ij}}{\partial A_{ij}} = C_{ij}$ expand i row ∴

若 A invertible/non-singular，$\frac{\partial \det A}{\partial A} = \det A(A^{-1})^T$　证 $A^{-1} = \frac{\text{adj}A}{\det A}$ 四.3

且 $\frac{\partial \ln\det A}{\partial A} = \frac{1}{\det A}\cdot\frac{\partial \det A}{\partial A} = (A^{-1})^T$ 且因 detA scalar func，$d\ln\det A = tr(A^{-1^T}dA)$

T_2 对 $f: \mathbb{R}^n \to \mathbb{R}^m$，$\vec{x} \in \mathbb{R}^n$，有 $\overset{m,1}{df} = \overset{m,n}{\frac{\partial f}{\partial x}}\overset{n,1}{dx}$

证 $LHS_j = \sum_i^n \frac{\partial f_j}{\partial x_i} dx_i$　$RHS_j = \sum_i^n \boxed{(\frac{\partial f}{\partial x})_{ji}} dx_i$　$\partial f_j/\partial x_i$

例4　求 $L = (a^Tx)(b^Tx)$ $\nabla_x L$ & $\nabla_x^2 L = H_L(x)$　证 由 ① 7.5.1，$\frac{\partial L}{\partial x} = (a^Tx)^T\frac{\partial b^Tx}{\partial x} + (b^Tx)^T\frac{\partial a^Tx}{\partial x}$　即是 scalar 也可 Transpose

$= x^Tab^T + x^Tba^T$，$\nabla_x L = \frac{\partial L}{\partial x}^T = (ba^T + ab^T)x$，$H_L(x) = \nabla\nabla_x L = (ba^T+ab^T)$

例5　$f: \mathbb{R}^2 \to \mathbb{R}$，$\binom{x_1}{x_2} \mapsto \frac{x_1^2}{6} + \frac{x_2^2}{4}$，$g: \mathbb{R} \to \mathbb{R}^2$，$t \mapsto \begin{bmatrix}3t+5\\2t-6\end{bmatrix}$ 求 $\frac{\partial f(g(t))}{\partial t}$ 证 由 ② 3. $\frac{\partial f(g(t))}{\partial g}\frac{\partial g}{\partial t} = \underbrace{\left(\frac{3t+5}{3}, \frac{2t-6}{2}\right)}_{\frac{\partial f}{\partial g} = (x_1/3,\ x_2/2)}\binom{3}{2}$

$f(\vec{x}) = \frac{x_1x_2}{2}$，$g(s,t) = \begin{bmatrix}4s+3t\\2s+t\end{bmatrix}$　$\frac{\partial}{\partial s}f(g(s,t)) = \frac{\partial f(g(s,t))}{\partial g}\frac{\partial g(s,t)}{\partial s} = \left(\frac{2s+t}{2}, \frac{4s+3t}{2}\right)\binom{4}{2}$　$\frac{\partial}{\partial t}f(g(s,t)) = (\cdots)\binom{3}{1}$